Graph Theory
Exam #1 (Chapter 1)
SPRING 2014

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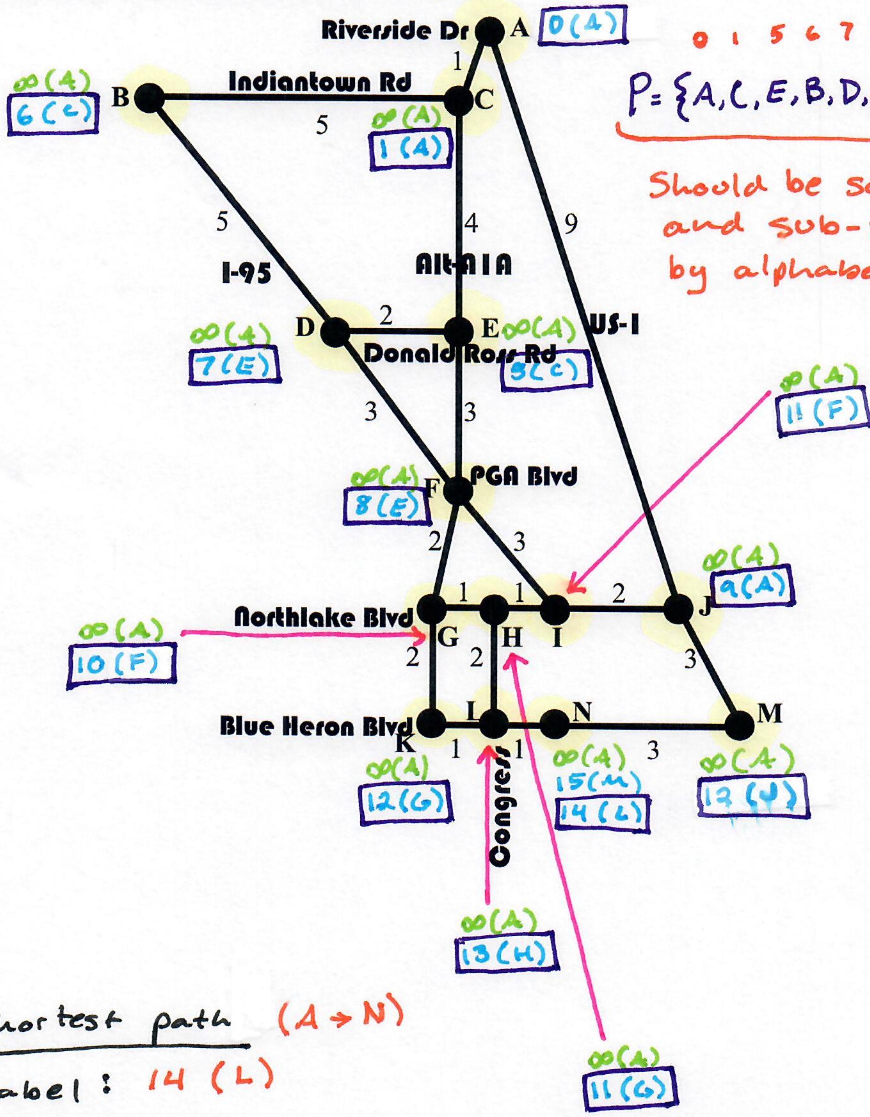
**SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT!!!**

1. Mr. Oddi needs help determining the shortest way to get to school in the morning, but he can't trust his navigation system. Using Dijkstra's algorithm, determine the shortest path with distance that he should take from his house (Vertex A) to Suncoast (Vertex N) in the morning. (Note: the shortest path is dependent on distance, not travel time.) Complete the algorithm to permanently label all vertices, and report the shortest path.




0 1 5 6 7 8 9 10 11 11 12 12 13 14

P = {A, C, E, B, D, F, J, G, H, I, K, M, L, N}

Should be sorted by distance and sub-sorted (ties broken) by alphabetization.

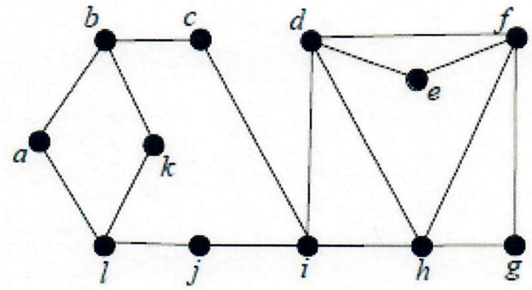
Finding a shortest path (A → N)

- Find its label: 14 (L)
- Trace backwards using predecessor until you reach node A.

N ← L ← H ← G ← F ← E ← C ← A

Shortest Path from A → N: $\boxed{\text{A → C → E → F → G → H → L → N, Length 14}}$

2. Given the following graph:




a. Write a valid path for each of the following (if it does not exist, state why):

i. Trail

$a \to b \to c$

ii. Trivial walk

$a$

iii. Simple path

$a \to b \to K \to l$

iv. Euler circuit

DNE – vertex b has degree 3 (odd)

v. Hamiltonian Circuit

DNE – vertex I breaks graph into separate components

b. Construct the adjacency matrix for the above graph, and use it to determine how many walks of length 3 there are from vertex D to vertex F.

$$\begin{array}{c} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h\\ i\\ j\\ K\\ l \end{array}
\begin{bmatrix}
0&1&0&0&0&0&0&0&0&0&0&1\\
1&0&1&0&0&0&0&0&0&0&1&0\\
0&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&1&0&1&1&0&0&0\\
0&0&0&1&0&1&0&0&0&0&0&0\\
0&0&0&1&1&0&1&1&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&0&0\\
0&0&0&1&0&1&1&0&1&0&0&0\\
0&0&1&1&0&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&1\\
1&0&0&0&0&0&0&0&0&1&1&0
\end{bmatrix} = A$$

$\begin{array}{cccccccccccc} a&b&c&d&e&f&g&h&i&j&K&l \end{array}$

3

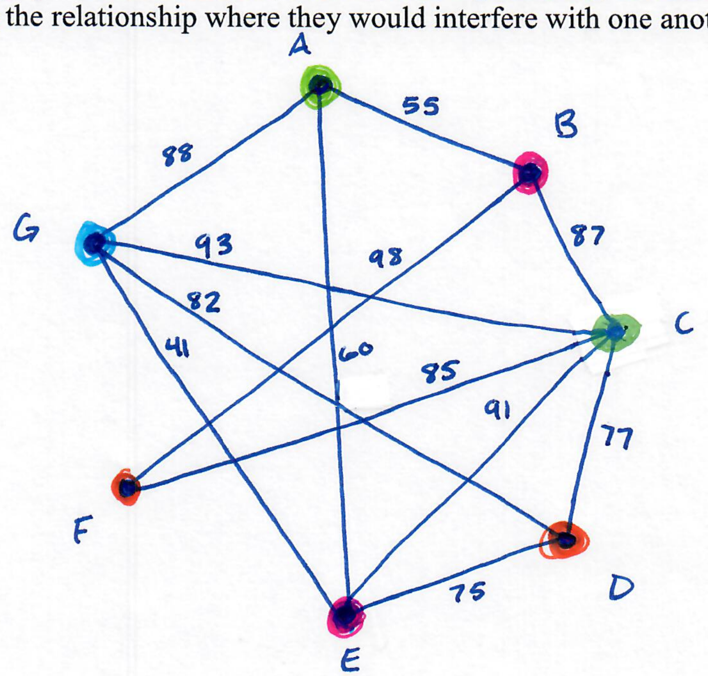
$A^3_{(3,5)} = \boxed{9}$

9 paths of length 3 from D to F.

3. The following table gives the distance between radio stations in miles. If two stations are within 100 miles of each other, their signals interfere with each other on the same frequency.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| A | * | 55 | 110 | 108 | 60 | 150 | 88 |
| B | 55 | * | 87 | 142 | 133 | 98 | 139 |
| C | 110 | 87 | * | 77 | 91 | 85 | 93 |
| D | 108 | 142 | 77 | * | 75 | 114 | 82 |
| E | 60 | 133 | 91 | 75 | * | 107 | 41 |
| F | 150 | 98 | 85 | 114 | 107 | * | 123 |
| G | 88 | 139 | 93 | 82 | 41 | 123 | * |

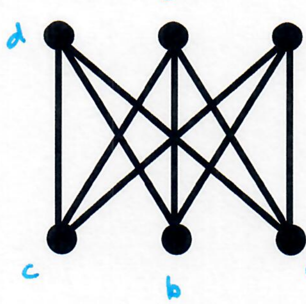
a. Construct a graph for this scenario, with vertices representing the different stations and edges representing the relationship where they would interfere with one another on the same frequency.




b. Color your graph to determine the minimum number of radio frequencies needed to avoid interference. What is the chromatic number?

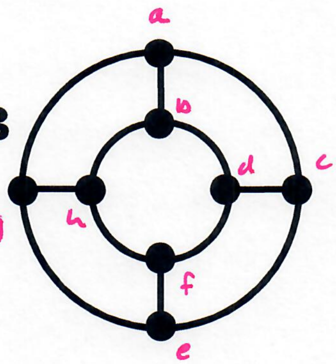
$\boxed{4}$ (vertices G, C, D, E form a complete subgraph)

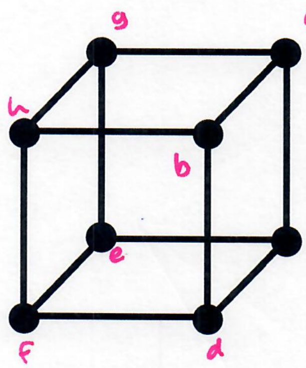
4. For the following six graphs, state whether any of the graphs are isomorphic to each other. If they are isomorphic, give a function to define the isomorphism. If they are not isomorphic to any of the others, state why this is the case.




A

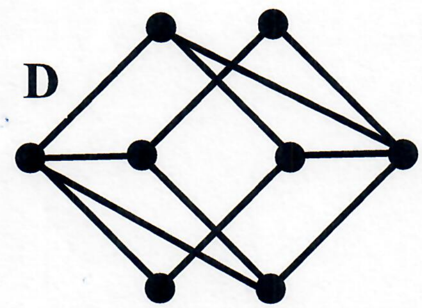
B

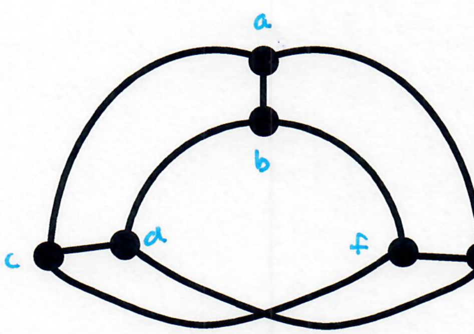



C

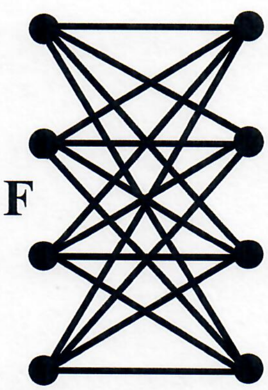
D






E

F



$C \to B$

$F(C_x) = B_x$

$A \to E$

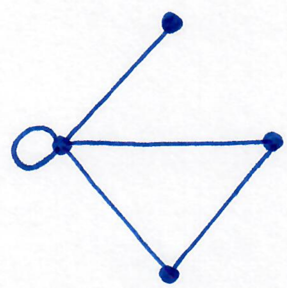
$F(A_x) = E_x$

D and F are not isomorphic, as all vertices in graph F have degree 4, and this is not the case for graph D.

5. Answer each of the following, justifying your answers appropriately.
   a. Consider a certain connected graph has 68 vertices and 72 edges. Does it have a circuit?

Yes. A connected graph with $n$ vertices and no cycles can have at most $n-1$ edges (tree).

   b. Consider a certain graph with four vertices of degrees 1, 2, 2, and 5. Draw it if it exists; otherwise, explain why it doesn't exist.

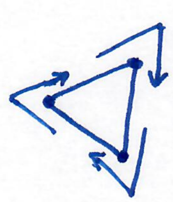


   c. Consider a certain graph with five vertices of degrees 1, 1, 1, 1, and 5. Draw it if it exists; otherwise, explain why it doesn't exist.

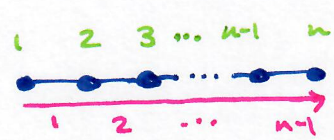
$1+1+1+1+5 = 9 \equiv 1 \mod 2$

$\therefore$ $\boxed{\text{DNE}}$ (sum of degrees must be even)

   d. In a multigraph with $n$ vertices, what is the maximum length of a **simple path**?



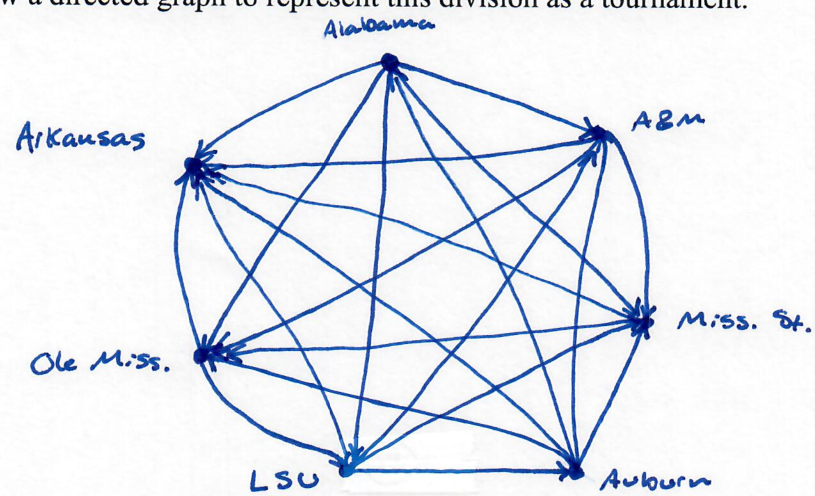
$\boxed{n-1}$, as the path can not cycle back to beginning vertex.



6. Consider the teams in the SEC West: Alabama, Arkansas, Auburn, LSU, Mississippi State, Ole Miss, and Texas A&M. During the 2013 NCAAF Season, each team played the other team once, with the scores of each game shown below in the table.

| | | |
|---|---|---|
| **Week 3**<br>Alabama (49) vs. Texas A&M (42)<br>Mississippi State (20) vs. Auburn (24)<br><br>**Week 4**<br>Auburn (21) vs. LSU (35)<br><br>**Week 5**<br>Ole Miss (0) vs. Alabama (25)<br>Texas A&M (45) vs. Arkansas (33)<br><br>**Week 6**<br>Ole Miss (22) vs. Auburn (30)<br>LSU (59) vs. Mississippi State (26) | **Week 7**<br>Texas A&M (41) vs. Ole Miss (38)<br><br>**Week 8**<br>Texas A&M (41) vs. Auburn (45)<br>LSU (24) vs. Ole Miss (27)<br>Arkansas (0) vs. Alabama (52)<br><br>**Week 10**<br>Auburn (35) vs. Arkansas (17)<br><br>**Week 11**<br>Arkansas (24) vs. Ole Miss (34)<br>Mississippi State (41) vs. Texas A&M (51)<br>LSU (17) vs. Alabama (38) | **Week 12**<br>Alabama (20) vs. Mississippi State (7)<br><br>**Week 13**<br>Mississippi State (24) vs. Arkansas (17)<br>Texas A&M (10) vs. LSU (34)<br><br>**Week 14**<br>Ole Miss (10) vs. Mississippi State (17)<br>Arkansas (27) vs. LSU (31)<br>Alabama (28) vs. Auburn (34) ☺ |

a. Draw a directed graph to represent this division as a tournament.




b. Construct the adjacency matrix for this tournament, showing how it provides the outdegrees and indegrees of the vertices; use these degrees to rank the seven teams.

$$\begin{array}{r} \text{Alabama} \\ \text{A\&M} \\ \text{Miss. St.} \\ \text{Auburn} \\ \text{LSU} \\ \text{Ole Miss} \\ \text{Arkansas} \end{array} \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{c} \underline{\text{outdegree}} \\ 5 \\ 3 \\ 2 \\ 5 \\ 4 \\ 2 \\ 0 \end{array}$$

Rankings

1. Auburn } TIE
2. Alabama
3. LSU
4. A&M
5. Miss. State } TIE
6. Ole Miss
7. Arkansas

**ANSWERS:**

1. Shortest Path = 14 (ACEFGHLN or ACEFGKLN)
   Labels: A-0(-), B-6(C), C-1(A), D-7(E), E-5(C), F-8(E), G-10(F), H-11(G), I-11(F) or 11(J), J-9(A), K-12(G), L-13(H) or 13(K), M-12(J), N-14(L)
2. a) i)-iii) answers may vary; iv) DNE – vertex B has odd degree; v) DNE – vertex I
   b) 9 walks of length 3
3. b) n = 4 (notice largest complete graph in part (a) is $K_4$)
4. A & E are isomorphic; B & C are isomorphic; D & F are not isomorphs (degree sequence doesn't match)
5. a) Yes – circuit has to use each edge exactly once and can repeat vertices
   b) if multigraph allowed, then it can be drawn; if specifically a graph, then DNE
   c) DNE – total degree = 9 (can't equal 2e)
   d) since a simple path has no repeated vertex, maximum length = n – 1
6. b)

$$\begin{array}{cc} \begin{array}{ccccccc}\text{Bama}&\text{Ark}&\text{AU}&\text{LSU}&\text{MS}&\text{OM}&\text{TA\&M}\end{array} & \text{outdegree} \\ \begin{bmatrix} 0&1&0&1&1&1&1\\ 0&0&0&0&0&0&0\\ 1&1&0&0&1&1&1\\ 0&1&1&0&1&0&1\\ 0&1&0&0&0&1&0\\ 0&1&0&1&0&0&0\\ 0&1&0&0&1&1&0 \end{bmatrix} & \begin{array}{c}5\\0\\5\\4\\2\\2\\3\end{array} \end{array}$$

   Rankings: 1-Auburn & Alabama, 2-LSU, 3- Texas A&M, 4-Miss St & Ole Miss, 5-Arkansas