4.1 Direct Proof and Counterexample I: Introduction

p. 161 # 1-3, 5, 7, 9-15, 17-18, 26-29, 35, 46, 48-49

1. a) $-17 = 2(-9) + 1$ Yes.
   b) 0 = 2(0) Yes.
   c) $2k - 1 = 2(k-1) + 1$ Yes.
2. a) $6m + 8n = 2(3m + 4n)$ Yes.
   b) $10mn + 7 = 2(5mn + 3) + 1$ Yes.
   c) $m^2 - n^2 = (m+n)(m-n)$ Yes. (divisible by $(m+n)$ and $(m-n)$).
3. a) $4rs = 2(2rs)$ Yes.
   b) $6r + 4s^2 + 3 = 2(3r + 2s^2 + 1) + 1$ Yes.
   c) $r^2 + 2rs + s^2 = (r+s)^2$ Yes. (divisible by $(r+s)$)
5. Let $m = 1$ and $n = -1$. $\frac{1}{1} + \frac{1}{-1} = 0$, and 0 is an integer.
7. Let $x = 59$. $2^{59} = 576460752303423488$ and $59^{10} = 97656250000000000$. $2^{59} > 59^{10}$
9. $25 = 5^2, 16 = 4^2$, and $9 = 3^2$. $25 = 16 + 9$, thus there is a perfect square that can be written as a sum of other perfect squares.
10. Let $n = 3$. $2(9) - 5(3) + 2 = 5$, and 5 is prime.
11. Let $a = -1$ and $b = 1$. $-1 < 1$ but $(-1)^2 = 1 < 1 = (1)^2$. Disproven.
12. Let $n = 1$. $\frac{1-1}{2} = 0$, and 0 is even. Disproven.
13. Let $m = 0$ and $n = 1$. $2(0) + 1 = 1$, and 1 is odd, but 0 is not odd. Disproven.
14. Let $a = b = 0$. $(0+0)^2 = 0^2 + 0^2$. The statement holds.
    Let $a = 1$ and $b = 2$. $(1+2)^2 = 9 \neq 5 = 1^2 + 2^2$. The statement does not hold.
    The property is true for some integers and false for others.
15. Let $a = 1$ and $n = 0$. $-1^0 = -1 \neq 1 = (-1)^0$ The statement does not hold.
    Let $a = n = 1$. $-1^1 = (-1)^1$ The statement holds.
    The property is true for some integers and false for others.
17. $2 = 1^2 + 1^2$ $4 = 2^2$ $6 = 2^2 + 1^2 + 1^2$
    $8 = 2^2 + 2^2$ $10 = 3^2 + 1^2$ $12 = 2^2 + 2^2 + 2^2$
    $14 = 3^2 + 2^2 + 1^2$ $16 = 4^2$ $20 = 4^2 + 2^2$
    $22 = 3^2 + 3^2 + 2^2$ $24 = 4^2 + 2^2 + 2^2$
18. $1 - 1 + 11 = 11\ (prime)$ $4 - 2 + 11 = 13\ (prime)$ $9 - 3 + 11 = 17\ (prime)$
    $16 - 4 + 11 = 23\ (prime)$ $25 - 5 + 11 = 31\ (prime)$ $36 - 6 + 11 = 41\ (prime)$
    $49 - 7 + 11 = 53\ (prime)$ $64 - 8 + 11 = 67\ (prime)$ $81 - 9 + 11 = 83\ (prime)$
    $100 - 10 + 11 = 101\ (prime)$
26. Let $m = 2k$ and $n = 2j + 1$, where $k, j \in \mathbb{Z}$. By the definition of even and odd, m is even and n is odd. $m - n = 2k - 2j + 1 = 2(k - j) + 1$, which is odd by definition.
27. Let $m = 2k + 1$ and $n = 2j + 1$, where $k, j \in \mathbb{Z}$. By the definition of odd, m and n are odd.
    $m + n = 2k + 2j + 2 = 2(k + j + 1)$, which is even by definition.
28. Let $n = 2k + 1$, where $k \in \mathbb{Z}$. By the definition of odd, n is odd.
    $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd by definition.
29. Let $n = 2k + 1$, where $k \in \mathbb{Z}$. By the definition of odd, n is odd.
    $3n + 5 = 3(2k + 1) + 5 = 6k + 8 = 2(k + 4)$, which is even by definition.
35. $m^2 - 1 = (m+1)(m-1)$, thus $m^2 - 1$ is divisible by $m + 1$ and $m - 1$.

46. Let $m = 2k$, where $k \in \mathbb{Z}$. By the definition of even, m is even. Let $j \in \mathbb{Z}$.
$mj = 2k(j) = 2(kj)$, which is even by definition.
48. Let $m = 2k$ and $n = 2j$, where $k, j \in \mathbb{Z}$. By the definition of even, m and n are even.
$m - n = 2k - 2j = 2(k - j)$, which is even by definition.
49. Let $m = 2k + 1$ and $n = 2j + 1$, where $k, j \in \mathbb{Z}$. By the definition of odd, m and n are odd.
$m - n = 2k + 1 - 2j - 1 = 2(k - j)$, which is even by definition.