4.2 Direct Proof and Counterexample II: Rational Numbers

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12. a) any rational number
b) integers a and b
c) $\left(\frac{a}{b}\right)^2$
d) b^2
e) zero product property
f) r^2 is rational

13. a) For all real numbers r, if r is rational, then $-r$ is rational.
b) The statement is true. Suppose that r is any rational number. By the definition of rational, $r = \frac{a}{b}$ for some integers a and b with $b \neq 0$. $-r = \frac{-a}{b}$, and $-a$ is an integer, so $-r$ is rational by the definition of rational.

14. a) For all real numbers r, if r is rational, then r^2 is rational.
b) The statement is true. Suppose that r is any rational number. By the definition of rational, $r = \frac{a}{b}$ for some integers a and b with $b \neq 0$. $r^2 = \frac{a^2}{b^2}$, and a^2 and b^2 are integers, so r^2 is rational by the definition of rational.

15. Let $r = \frac{a}{b}$ and $s = \frac{c}{d}$ $\quad r, s \in \mathbb{Q} \quad a, b, c, d \in \mathbb{Z} \quad b \neq 0, d \neq 0$
$r \cdot s = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, which is rational by definition. The statement is true.

16. Let $r = \frac{1}{2}$ and $s = \frac{0}{1}$.
$r \div s = \frac{1}{2} \cdot \frac{1}{0} = \frac{1}{0} \notin \mathbb{Q}$. The statement is false.

17. Let $r = \frac{a}{b}$ and $s = \frac{c}{d}$ $\quad r, s \in \mathbb{Q} \quad a, b, c, d \in \mathbb{Z} \quad b \neq 0, d \neq 0$
$r - s = \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$, which is rational by definition. The statement is true.

18. Let $r = \frac{a}{b}$ and $s = \frac{c}{d}$ $\quad r, s \in \mathbb{Q} \quad a, b, c, d \in \mathbb{Z} \quad b \neq 0, d \neq 0$
$\frac{r+s}{2} = \frac{\frac{a}{b}+\frac{c}{d}}{2} = \frac{ad+bc}{2bd}$, which is rational by definition. The statement is true.

21. Because the product of two even integers is even, m^2 is even. (1)
Because the product of two odd integers is odd, $3n$ is odd. (3)
Because the sum of an even integer and an odd integer is odd, $m^2 + 3n$ is odd. (5)
The statement is true.

22. Because the product of two odd integers is odd, a^2 is odd. (3)
Because the sum of any two odd integers is even, $a^2 + a$ is even. (2)
The statement is false.

23. Because the sum of two even integers is even, $k + 2$ is even. (1)
Because the product of two even integers is even, $(k + 2)^2$ is even. (1)
Because the difference of two odd numbers is even, $m - 1$ is even. (2)
Because the product of two even numbers is even, $(m - 1)^2$ is even. (1)
Because the difference of any two even integers is even, $(k + 2)^2 - (m - 1)^2$ is even. (1)

25. $Let\ r = \frac{a}{b}$ $\quad r \in \mathbb{Q} \quad a, b \in \mathbb{Z} \quad b \neq 0$
$3r^2 - 2r + 4 = \frac{3a^2}{b^2} - \frac{2a}{b} + 4 = \frac{3a^2b-2ab^2+4b^3}{b^3}$, which is rational by definition.

26. $Let\ s = \frac{a}{b}$ $\quad s \in \mathbb{Q}$ $\quad a, b \in \mathbb{Z}$ $\quad b \neq 0$

$5s^3 + 8s^2 - 7 = \frac{5a^3}{b^3} + \frac{8a^2}{b^2} - 7 = \frac{(5a^3b^2 + 8a^2b^3 - 7b^5)}{b^5}$, which is rational by definition.

30. Let the quadratic equation $x^2 + bx + c = 0$ have rational solution r and second solution s.

$x^2 + bx + c = (x - r)(x - s) = x^2 - (r + s)x + rs$

$b = -r - s \qquad c = rs$

$s = -r - b \qquad s = \frac{c}{r}$

By the properties of rational numbers, s is rational.

35. The proof assumes what is to be proved.
36. The proof is arguing from example.
37. The proof uses the same variable for different purposes.
38. The proof jumps to conclusions.
39. The proof confuses the known and the unproven.