5.2 Mathematical Induction I

p.257 #7-9, 11-14, 19

(7) $\forall n \in \mathbb{Z} \mid n \geq 1,\ 1+6+11+\cdots+(5n-4) = \frac{n(5n-3)}{2}$

$$P(1) = 1 = \frac{(1)(5(1)-3)}{2} = \frac{1\cdot 2}{2} = 1 \checkmark$$

$$P(K+1) = \underbrace{1+6+11+\cdots+(5K-4)}+5(K+1)-4$$

$$\frac{K(5K-3)}{2} + 5K+5-4 = \frac{(K+1)(5(K+1)-3)}{2}$$

$$K(5K-3)+10K+2 = (K+1)(5K+2)$$

$$5K^2+7K+2 = 5K^2+7K+2) \checkmark$$

$\boxed{\text{True by M.I.}}$

(8) $\forall n \in \mathbb{Z} \mid n \geq 0,\ 1+2^1+2^2+\cdots+2^n = 2^{n+1}-1$

$$P(0) = 1 = 2^1 - 1 = 1 \checkmark$$

$$P(K+1) = \underbrace{1+2^1+2^2+\cdots+2^K}+2^{K+1} = 2^{K+2}-1$$

$$(2^{K+1}-1)+2^{K+1} = 2^{K+2}-1$$

$$2^1(2^{K+1})-1 = 2^{K+2}-1$$

$$2^{K+2}-1 = 2^{K+2}-1 \checkmark$$

$\boxed{\text{True by M.I.}}$

(9) $\forall n \in \mathbb{Z} \mid n \geq 3,\ 4^3+4^4+4^5+\cdots+4^n = \frac{4(4^n-16)}{3}$

$$P(3) = 4^3 = \frac{4(4^3-16)}{3} = \frac{4\cdot 48}{3} = 64 \checkmark$$

$$P(K+1) = \underbrace{4^3+4^4+4^5+\cdots+4^K}+4^{K+1} = \frac{4(4^{K+1}-16)}{3}$$

$$\frac{4(4^K-16)}{3}+4^{K+1} = \frac{4(4^{K+1}-16)}{3}$$

$$4(4^K-16)+3\cdot 4^{K+1} = 4(4^{K+1}-16)$$

$$4^K-16+3\cdot 4^K = 4^{K+1}-16$$

$$4\cdot 4^K = 4^{K+1}$$

$$4^{K+1} = 4^{K+1} \checkmark$$

$\boxed{\text{True by M.I.}}$

(11) $\forall n \in \mathbb{Z} \mid n \geq 1,\ 1^3+2^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

$$P(1) = 1^3 = 1 = \left[\frac{1(1+1)}{2}\right]^2 = \left(\frac{2}{2}\right)^2 = 1 \checkmark$$

$$P(K+1) = \underbrace{1^3+2^3+\cdots+K^3}+(K+1)^3 = \left[\frac{(K+1)(K+2)}{2}\right]^2$$

$$\left[\frac{K(K+1)}{2}\right]^2+(K+1)^3 = \frac{(K+1)^2(K+2)^2}{4}$$

$$\cancel{(K+1)^2}\left[\frac{K^2}{4}+(K+1)\right] = \cancel{(K+1)^2}\,\frac{(K+2)^2}{4}$$

$$K^2+4K+4 = (K+2)^2$$

$$K^2+4K+4 = K^2+4K+4 \checkmark$$

$\boxed{\text{True by M.I.}}$

(12) $\forall n \in \mathbb{Z} \mid n \geq 1,\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} = \frac{n}{n+1}$

$$P(1) = \frac{1}{2} = \frac{1}{1+1} = \frac{1}{2} \checkmark$$

$$P(K+1) = \underbrace{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{K(K+1)}}+\frac{1}{(K+1)(K+2)} = \frac{K+1}{K+2}$$

$$\frac{K}{K+1}+\frac{1}{(K+1)(K+2)} = \frac{(K+2)K}{(K+1)(K+2)}+\frac{1}{(K+1)(K+2)} = \frac{K+1}{K+2}$$

$$\frac{K^2+2K+1}{(K+1)(K+2)} \rightarrow \frac{\cancel{(K+1)}(K+1)}{\cancel{(K+1)}(K+2)} = \frac{K+1}{K+2} \checkmark$$

$\boxed{\text{True by M.I.}}$

(13) $\forall n \in \mathbb{Z} \mid n \geq 2,\ \sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$

$$P(2) = \sum_{i=1}^{2-1} i(i+1) = 1(1+1) = 2 = \frac{2(2-1)(2+1)}{3} = \frac{2\cdot 1\cdot 3}{3} = 2 \checkmark$$

$$P(K+1) = \underbrace{\sum_{i=1}^{K-1} i(i+1)}+K(K+1) = \frac{(K+1)(K)(K+2)}{3}$$

$$\frac{K(K-1)(K+1)}{3}+K(K+1) = \frac{(K+1)(K)(K+2)}{3}$$

$$K(K-1)(K+1)+3K(K+1) = K(K+1)(K+2)$$

$$K(K-1)+3K = K(K+2)$$

$$K^2-K+3K = K^2+2K$$

$$K^2+2K = K^2+2K$$

$\boxed{\text{True by M.I.}}$

(14) $\forall n \in \mathbb{Z} \mid n \geq 0, \sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$

$$P(0) = \sum_{i=1}^{0+1} i \cdot 2^i = 1 \cdot 2^1 = 2 = 0 \cdot 2^{0+2} + 2 = 2 \checkmark$$

$$P(K+1) = \sum_{i=1}^{K+1+1} i \cdot 2^i = \underbrace{\sum_{i=1}^{K+1} i \cdot 2^i}_{K(2^{K+2})+2} + (K+2) \cdot 2^{K+2} = (K+1) \cdot 2^{K+3} + 2$$

$$K(2^{K+2}) + 2 + (K+2) \cdot 2^{K+2} = (K+1) \cdot 2^{K+3} + 2$$

$$K \cdot 2^{K+2} + 2 + K \cdot 2^{K+2} + 2^{K+3} = (K+1) \cdot 2^{K+3} + 2$$

$$K \cdot 2^{K+3} + 2^{K+3} + 2 = (K+1) \cdot 2^{K+3} + 2$$

$$(K+1) \cdot 2^{K+3} + 2 = (K+1) \cdot 2^{K+3} + 2$$

$\boxed{\text{True by M.I.}}$

(19) $\forall n \in \mathbb{Z} \mid n \geq 1, \frac{d(x^n)}{dx} = n x^{n-1}$

$$P(1) = \frac{d(x^1)}{dx} = 1 = (1) x^{1-1} = 1 \checkmark$$

$$P(K+1) = \frac{d(x^{K+1})}{dx} = \frac{d(x \cdot x^K)}{dx} = (K+1) x^K$$

$$= x \underbrace{\frac{d(x^K)}{dx}} + x^K \frac{d(x)}{dx} = (K+1) x^K$$

$$= x(K \cdot x^{K-1}) + x^K(1) = (K+1) x^K$$

$$= K x^K + x^K = (K+1) x^K$$

$$= (K+1) x^K = (K+1) x^K$$

$\boxed{\text{True by M.I.}}$