$\boxed{\text{5.3 Mathematical Induction II}}$ p.266 #4, 9, 11, 12, 15, 24, 28

(4) $\forall n \in \mathbb{Z} \mid n \geq 1,\ 1-4+9-\cdots+(-1)^{n-1} n^2 = (-1)^{n-1}(1+2+\cdots+n)$

$P(1):\quad 1 = (-1)^{1-1}(1) = 1 \cdot 1 = 1$ ✓

$P(K+1):\ \underbrace{1-4+9-\cdots+(-1)^{K-1}K^2}_{(-1)^{K-1}(1+2+\cdots+K)} + (-1)^K (K+1)^2 = (-1)^K(1+2+\cdots+K+(K+1))$

$(-1)^{K-1}(1+2+\cdots+K)+(-1)^K(K+1)^2 = (-1)^K(1+2+\cdots+K+(K+1))$

$-(1+2+\cdots+K)+(K+1)^2 = (1+2+\cdots+K+(K+1))$

$-\frac{K(K+1)}{2}+(K+1)^2 = \frac{K(K+1)}{2} + K+1$

$-K^2-K+2K^2+4K+2 = K^2+K+2K+2$

$K^2+3K+2 = K^2+3K+2$ ✓

$\boxed{\text{True by M.I.}}$

(9) $\forall n \in \mathbb{Z} \mid n \geq 0,\ 6 \mid 7^n - 1$

$P(0):\ 6 \mid 7^0 - 1$

$6 \mid 1-1$

$6 \mid 0$ ✓

$P(K+1) = 6 \mid 7^{K+1} - 1$

$6 \mid 7(7^K) - 1$

Let $r = 7^K - 1$

$6 \mid r$ by the inductive hypothesis

$6 \mid 7(r+1) - 1$

$7^K = r+1$

$6 \mid 7r - 6$

$6 \mid r \rightarrow 6 \mid 7r$

$6 \mid -6$

$\therefore 6 \mid 7r - 6 \longrightarrow 6 \mid 7^n - 1\ \forall n \in \mathbb{Z} \mid n \geq 0$ by Def. of Divisibility

$\boxed{\text{True by M.I.}}$

(11) $\forall n \in \mathbb{Z} \mid n \geq 0,\ 8 \mid 3^{2n} - 1$

$P(0):\ 8 \mid 3^0 - 1 = 8 \mid 0$ ✓

$P(K+1):\ 8 \mid 3^{2K+2} - 1$

Let $r = 3^{2K} - 1$

$8 \mid r$ by the inductive hypothesis

$8 \mid 9 \cdot 3^{2K} - 1$

$3^{2K} = r+1$

$8 \mid 9(r+1) - 1$

$8 \mid 9r + 8$

$8 \mid r \rightarrow 8 \mid 9r$

$8 \mid 8$

$\therefore 8 \mid 9r + 8 \longrightarrow 8 \mid 3^{2n} - 1\ \forall n \in \mathbb{Z} \mid n \geq 0$ by Def. of Divisibility

$\boxed{\text{True by M.I.}}$

(12) $\forall n \in \mathbb{Z} \mid n \geq 0,\ 5 \mid 7^n - 2^n$

$P(0): 5 \mid 7^0 - 2^0 = 5 \mid 0$ ✓

$P(K+1): 5 \mid 7^{K+1} - 2^{K+1}$

Let $r = 7^K - 2^K$

$5 \mid r$ by the inductive hypothesis

$5 \mid 7 \cdot 7^K - 2 \cdot 2^K$

$5 \mid 5 \cdot 7^K + 2 \cdot 7^K - 2 \cdot 2^K$

$5 \mid 5 \cdot 7^K + 2(7^K - 2^K)$

$5 \mid 5 \cdot 7^K + 2r$

$5 \mid 5 \rightarrow 5 \mid 5 \cdot 7^K$

$5 \mid r \rightarrow 5 \mid 2r$

$\therefore 5 \mid 5 \cdot 7^K - 2r$

$\therefore 5 \mid 7^n - 2^n\ \forall n \in \mathbb{Z} \mid n \geq 0$ by Def. of Divisibility $\boxed{\text{True by M.I.}}$

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(15) $\forall n \in \mathbb{Z} \mid n \geq 0,\ 6 \mid n(n^2 + 5)$

$P(0): 6 \mid 0(0^2 + 5) = 6 \mid 0$ ✓

$P(K+1): 6 \mid (K+1)((K+1)^2 + 5) = 6 \mid (K+1)(K^2 + 2K + 6) = 6 \mid K^3 + 3K^2 + 8K + 6$

$= 6 \mid K(K^2 + 5) + 3K^2 + 3K + 6$

Let $r = K(K^2 + 5)$ $6 \mid r$ by the I.H.

$= 6 \mid K(K^2 + 5) + 3(K^2 + K + 2)$

$= 6 \mid r + 3(K^2 + K + 2)$

$6 \mid r$ If $2 \mid K^2 + K + 2$, then $3(K^2 + K + 2) = 6j$ for some integer K.

$K^2 + K + 2 = K(K+1) + 2$

the sum of two even integers is even

the product of two consecutive integers is even

$\therefore 2 \mid K^2 + K + 2 \rightarrow 6 \mid r + 3(K^3 + K + 2) \rightarrow 6 \mid n(n^2 + 5)\ \forall n \in \mathbb{Z} \mid n \geq 0$

by Def. of Divisibility $\boxed{\text{True by M.I.}}$

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(24) $a_1 = 3 \quad a_k = 7a_{k-1} \ \forall\ k \in \mathbb{Z} \mid k \geq 2.$

I.H.: $\forall n \in \mathbb{Z} \mid n \geq 1,\ a_n = 3 \cdot 7^{n-1}$ $\qquad$ $P(1): a_1 = 3 = 3 \cdot 7^{1-1} = 3 \cdot 1 = 3$ ✓

$P(k+1)$: $a_{k+1} = 3 \cdot 7^k$ $\qquad$ by seq. definition, $a_{k+1} = 7a_k$

$a_{k+1} = 3 \cdot 7^{k-1} \cdot 7^1$

$a_{k+1} = a_k \cdot 7 = 7a_k$ ✓ matches seq. definition

$\therefore$ $\boxed{\text{True by M. I.}}$

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(28) $\forall n \in \mathbb{Z} \mid n \geq 1,\ \dfrac{1+3+\cdots+(2n-1)}{(2n+1)+(2n+3)+\cdots+(2n+2n-1)} = \dfrac{1}{3}$ $\qquad$ $P(1): \dfrac{1}{3} = \dfrac{2(1)-1}{(2\cdot1+2\cdot1-1)} = \dfrac{1}{3}$

$1+3+\cdots+(2n-1) = \sum_{i=1}^{n}(2i-1) = n^2$ $\qquad$ $\dfrac{n^2}{3n^2} = \dfrac{1}{3}$ ✓

$(2n+1)+(2n+3)+\cdots+(4n-1) = \sum_{i=1}^{2n}(2i+2i-1) = 3n^2$ $\qquad$ $\boxed{\text{True by M. I.}}$