5.6 Defining Sequences Recursively

p.302 #3,5,7,10-11,14-15,22-27,38,40

(3)
$c_0 = 1$
$c_1 = 1(1)^2 = 1$
$c_2 = 2(1)^2 = 2$
$c_3 = 3(2)^2 = 12$
$c_4 = 4(12)^2 = 576$

(5)
$s_0 = 1$
$s_1 = 1$
$s_2 = 1 + 2(1) = 3$
$s_3 = 3 + 2(1) = 5$
$s_4 = 5 + 2(3) = 11$

(7)
$u_1 = 1$
$u_2 = 1$
$u_3 = 3(1) - 1 = 2$
$u_4 = 4(2) - 1 = 7$
$u_5 = 5(7) - 2 = 33$

(10) $\forall k \in \mathbb{Z} \mid k \geq 1,\ b_k = 4b_{k-1}$ $\quad b_n = 4^n$

$P(0)$: $b_0 = 1 = 4^0 = 1$ ✓

$P(n+1)$: $b_{n+1} = 4b_n$ $\quad b_n = 4^n$

$b_{n+1} = 4 \cdot 4^n$

$b_{n+1} = 4^{n+1}$ ✓ $\boxed{\text{True by M.I.}}$

(11) $\forall k \in \mathbb{Z} \mid k \geq 1,\ c_k = 2c_{k-1} + 1$ $\quad c_n = 2^n - 1$

$P(0)$: $c_0 = 2^0 - 1 = 0$

$P(1)$: $c_1 = 2^1 - 1 = 1 = 2c_0 + 1 = 0 + 1 = 1$ ✓

$P(n+1)$: $c_{n+1} = 2c_n + 1$ $\quad c_n = 2^n - 1$

$c_{n+1} = 2(2^n - 1) + 1$

$c_{n+1} = 2^{n+1} - 2 + 1$

$c_{n+1} = 2^{n+1} - 1$ ✓ $\boxed{\text{True by M.I.}}$

(14) $\forall k \in \mathbb{Z} \mid k \geq 2,\ d_k = 5d_{k-1} - 6d_{k-2}$ $\quad d_n = 3^n - 2^n$

$P(2)$: $d_2 = 5d_1 - 6d_0$ $\quad d_1 = 3^1 - 2^1 = 1$

$d_2 = 5 - 0 = 5$ $\quad d_0 = 3^0 - 2^0 = 0$

$d_2 = 3^2 - 2^2 = 5$ ✓

$P(n+1)$: $d_{n+1} = 5d_n - 6d_{n-1}$ $\quad d_{n-1} = 3^{n-1} - 2^{n-1}$

$d_{n+1} = 5 \cdot 3^n - 5 \cdot 2^n - 6 \cdot 3^{n-1} + 6 \cdot 2^{n-1}$

$d_{n+1} = 5 \cdot 3^n - 5 \cdot 2^n - 2 \cdot 3^n + 3 \cdot 2^n$

$d_{n+1} = 3 \cdot 3^n - 2 \cdot 2^n = 3^{n+1} - 2^{n+1}$ ✓

$\boxed{\text{True by M.I.}}$

(15) $C_n = \frac{1}{n+1}\binom{2n}{n}$ by definition

$$C_n = \frac{(2n)!}{(n+1)\,n!\,n!}$$

Show: $\frac{1}{4n+2}\binom{2n+2}{n+1} = C_n$

$$\frac{1\,(2n+2)(2n+1)(2n)!}{(4n+2)(n+1)\,n!\,(n+1)\,n!}$$

$$\frac{\cancel{2}\cancel{(n+1)}\cancel{(2n+1)}(2n)!}{\cancel{2}\cancel{(2n+1)}\cancel{(n+1)}\,n!\,(n+1)\,n!} = \frac{(2n)!}{(n+1)\,n!\,n!}$$ ✓

(22) a) $r_n = r_{n-1} + 4r_{n-2}$ $\forall n \geq 1$ $\quad r_0 = 1, r_1 = 1$

b)

n	r_n
0	1
1	1
2	5
3	9
4	29
5	65
6	181

c) 99322 rabbits

(49661 pairs)

(23) a) $S_n = S_{n-1} + 3S_{n-3}$ $\quad \forall n \geq 1$ $\quad S_0 = S_1 = S_2 = 1$

b)

n	S_n
0	1
1	1
2	1
3	4
4	7
5	10

c) 1808 rabbits

(904 pairs)

(24) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, $\boxed{233, \ 377}$

(25) a) Any term of F is the sum of the two previous terms. F_{k-1} and F_k are the terms preceeding F_{k+1}, so the equation is valid.

b) $F_{k+2} = F_{k+1} + F_k$

c) $F_{k+3} = F_{k+2} + F_{k+1}$

26) $F_k = 3F_{k-3} + 2F_{k-4} \quad \forall k \in \mathbb{Z} \mid k \geq 4$

$$F_{k-1} + F_{k-2} = 3F_{k-3} + 2F_{k-4}$$

$$F_{k-2} + F_{k-3} + F_{k-3} + F_{k-4} = 3F_{k-3} + 2F_{k-4}$$

$$F_{k-2} + 2F_{k-3} + F_{k-4} = 3F_{k-3} + 2F_{k-4}$$

$$F_{k-2} = F_{k-3} + F_{k-4}$$ ✓ valid

27) $F_k^2 - F_{k-1}^2 = F_k F_{k-1} - F_{k+1} F_{k-1} \quad \forall k \in \mathbb{Z} \mid k \geq 1$

False. Textbook error.

38) a) $S_n = (1.0025) S_{n-1}$

$S_0 = 10000$

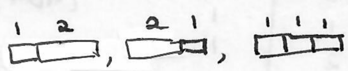
b) $S_{12} = \$10304.16$

c) 3.0416%

40) $t_0 = 1$ (only one way to construct: no blocks)

$t_1 = 1$ (one 1 block)

$t_2 = 2$ (one two block, two one blocks)

$t_3 = 3$ ()

$t_n = t_{n-4} + t_{n-2} + t_{n-1} \quad \forall n \in \mathbb{Z} \mid n \geq 4$

Exp: When building a tower of height n, we can place any of the three blocks as long as there is a height of at least 4 to go (hence the restriction $n \geq 4$). If we place the 4 block, we must now consider the number of ways to build a tower of size n-4 (hence t_{n-4}), same for the 2 and 1 blocks. This gives the recurrence relation above.