Chapter 1

Systems of Linear Equations and Matrices

Section 1.1

Exercise Set 1.1

1. (a), (c), and (f) are linear equations in x_1, x_2, and x_3.
(b) is not linear because of the term x_1x_3.
(d) is not linear because of the term x^{-2}.
(e) is not linear because of the term $x_1^{3/5}$.

3. (a) and (d) are linear systems.
(b) is not a linear system because the first and second equations are not linear.
(c) is not a linear system because the second equation is not linear.

5. By inspection, (a) and (d) are both consistent; $x_1 = 3$, $x_2 = 2$, $x_3 = -2$, $x_4 = 1$ is a solution of (a) and $x_1 = 1$, $x_2 = 3$, $x_3 = 2$, $x_4 = 2$ is a solution of (d). Note that both systems have infinitely many solutions.

7. (a), (d), and (e) are solutions.
(b) and (c) do not satisfy any of the equations.

9. **(a)** $7x - 5y = 3$

$$x = \frac{5}{7}y + \frac{3}{7}$$

Let $y = t$. The solution is

$$x = \frac{5}{7}y + \frac{3}{7}$$
$$y = t$$

(b) $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$

$$x_1 = \frac{1}{4}x_2 - \frac{5}{8}x_3 + \frac{3}{4}x_4 - \frac{1}{8}$$

Let $x_2 = r$, $x_3 = s$, and $x_4 = t$. The solution is

$$x_1 = \frac{1}{4}r - \frac{5}{8}s + \frac{3}{4}t - \frac{1}{8}$$
$$x_2 = r$$
$$x_3 = s$$
$$x_4 = t$$

11. **(a)** $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ corresponds to

$$\begin{aligned} 2x_1 \qquad &= 0 \\ 3x_1 - 4x_2 &= 0. \\ x_2 &= 1 \end{aligned}$$

(b) $\begin{bmatrix} 3 & 0 & -2 & 5 \\ 7 & 1 & 4 & -3 \\ 0 & -2 & 1 & 7 \end{bmatrix}$ corresponds to

$$\begin{aligned} 3x_1 \quad\quad - 2x_3 &= 5 \\ 7x_1 + x_2 + 4x_3 &= -3. \\ -2x_2 + x_3 &= 7 \end{aligned}$$

(c) $\begin{bmatrix} 7 & 2 & 1 & -3 & 5 \\ 1 & 2 & 4 & 0 & 1 \end{bmatrix}$ corresponds to

$$\begin{aligned} 7x_1 + 2x_2 + x_3 - 3x_4 &= 5 \\ x_1 + 2x_2 + 4x_3 \quad\quad &= 1 \end{aligned}.$$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ corresponds to

$$\begin{aligned} x_1 &= 7 \\ x_2 &= -2 \\ x_3 &= 3 \\ x_4 &= 4 \end{aligned}.$$

13. **(a)** The augmented matrix for $\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}$

is $\begin{bmatrix} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{bmatrix}$.

(b) The augmented matrix for

$\begin{aligned} 6x_1 - x_2 + 3x_3 &= 4 \\ 5x_2 - x_3 &= 1 \end{aligned}$ is $\begin{bmatrix} 6 & -1 & 3 & 4 \\ 0 & 5 & -1 & 1 \end{bmatrix}$.

(c) The augmented matrix for

$$\begin{aligned} 2x_2 \quad\quad - 3x_4 + x_5 &= 0 \\ -3x_1 - x_2 + x_3 \quad\quad\quad\quad &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{aligned}$$

is $\begin{bmatrix} 0 & 2 & 0 & -3 & 1 & 0 \\ -3 & -1 & 1 & 0 & 0 & -1 \\ 6 & 2 & -1 & 2 & -3 & 6 \end{bmatrix}$.

(d) The augmented matrix for $x_1 - x_5 = 7$ is $[1 \quad 0 \quad 0 \quad 0 \quad -1 \quad 7]$.

15. If (a, b, c) is a solution of the system, then $ax_1^2 + bx_1 + c = y_1$, $ax_2^2 + bx_2 + c = y_2$, and $ax_3^2 + bx_3 + c = y_3$ which simply means that the points are on the curve.

17. The solutions of $x_1 + kx_2 = c$ are $x_1 = c - kt$, $x_2 = t$ where t is any real number. If these satisfy $x_1 + lx_2 = d$, then $c - kt + lt = d$ or $c - d = (k - l)\,t$ for all real numbers t. In particular, if $t = 0$, then $c = d$, and if $t = 1$, then $k = l$.

True/False 1.1

(a) True; $x_1 = x_2 = \cdots = x_n = 0$ will be a solution.

(b) False; only multiplication by nonzero constants is acceptable.

(c) True; if $k = 6$ the system has infinitely many solutions, while if $k \neq 6$, the system has no solution.

(d) True; the equation can be solved for one variable in terms of the other(s), yielding parametric equations that give infinitely many solutions.

(e) False; the system $\begin{aligned} 3x - 5y &= -7 \\ 2x + 9y &= 20 \\ 6x - 10y &= -14 \end{aligned}$ has the solution $x = 1$, $y = 2$.

(f) False; multiplying an equation by a nonzero constant c does not change the solutions of the system.

(g) True; subtracting one equation from another is the same as multiplying an equation by -1 and adding it to another.

(h) False; the second row corresponds to the equation $0x_1 + 0x_2 = -1$ or $0 = -1$ which is false.

Section 1.2

Exercise Set 1.2

1. **(a)** The matrix is in both row echelon and reduced row echelon form.

(b) The matrix is in both row echelon and reduced row echelon form.

(c) The matrix is in both row echelon and reduced row echelon form.

(d) The matrix is in both row echelon and reduced row echelon form.

(e) The matrix is in both row echelon and reduced row echelon form.

(f) The matrix is in both row echelon and reduced row echelon form.

(g) The matrix is in row echelon form.

3. (a) The matrix corresponds to the system

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 7 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 5 \end{aligned} \quad \text{or} \quad \begin{aligned} x_1 &= 7 + 3x_2 - 4x_3 \\ x_2 &= 2 - 2x_3 \\ x_3 &= 5 \end{aligned}.$$

Thus $x_3 = 5$, $x_2 = 2 - 2(5) = -8$, and $x_1 = 7 + 3(-8) - 4(5) = -37$. The solution is $x_1 = -37$, $x_2 = -8$, $x_3 = 5$.

(b) The matrix corresponds to the system

$$\begin{aligned} x_1 \qquad + 8x_3 - 5x_4 &= 6 \\ x_2 + 4x_3 - 9x_4 &= 3 \\ x_3 + x_4 &= 2 \end{aligned} \quad \text{or}$$

$$\begin{aligned} x_1 &= 6 - 8x_3 + 5x_4 \\ x_2 &= 3 - 4x_3 + 9x_4 \\ x_3 &= 2 - x_4 \end{aligned}.$$

Let $x_4 = t$, then $x_3 = 2 - t$, $x_2 = 3 - 4(2 - t) + 9t = 13t - 5$, and $x_1 = 6 - 8(2 - t) + 5t = 13t - 10$. The solution is $x_1 = 13t - 10$, $x_2 = 13t - 5$, $x_3 = -t + 2$, $x_4 = t$.

(c) The matrix corresponds to the system

$$\begin{aligned} x_1 + 7x_2 - 2x_3 \qquad - 8x_5 &= -3 \\ x_3 + x_4 + 6x_5 &= 5 \\ x_4 + 3x_5 &= 9 \end{aligned} \quad \text{or}$$

$$\begin{aligned} x_1 &= -3 - 7x_2 + 2x_3 + 8x_5 \\ x_3 &= 5 - x_4 - 6x_5 \\ x_4 &= 9 - 3x_5 \end{aligned}$$

Let $x_2 = s$ and $x_5 = t$, then $x_4 = 9 - 3t$, $x_3 = 5 - (9 - 3t) - 6t = -3t - 4$, and $x_1 = -3 - 7s + 2(-3t - 4) + 8t = -7s + 2t - 11$. The solution is $x_1 = -7s + 2t - 11$, $x_2 = s$, $x_3 = -3t - 4$, $x_4 = -3t + 9$, $x_5 = t$.

(d) The last line of the matrix corresponds to the equation $0x_1 + 0x_2 + 0x_3 = 1$, which is not satisfied by any values of x_1, x_2, and x_3, so the system is inconsistent.

5. The augmented matrix is $\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$.

Add the first row to the second row and add −3 times the first row to the third row.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$

Multiply the second row by −1.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$

Add 10 times the second row to the third row.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix}$$

Multiply the third row by $-\frac{1}{52}$.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Add 5 times the third row to the second row and −2 times the third row to the first row.

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Add −1 times the second row to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution is $x_1 = 3$, $x_2 = 1$, $x_3 = 2$.

7. The augmented matrix is

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}.$$

Add −2 times the first row to the second row, 1 times the first row to the third row, and −3 times the first row to the fourth row.

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$$

Multiply the second row by $\frac{1}{3}$ and then add −1 times the new second row to the third row and add −3 times the new second row to the fourth row.

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Add the second row to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x \qquad - w &= -1 \\ y - 2z \qquad &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x &= w - 1 \\ y &= 2z \end{aligned}.$$

Let $z = s$ and $w = t$. The solution is $x = t - 1$, $y = 2s$, $z = s$, $w = t$.

9. In Exercise 5, the following row echelon matrix occurred.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ x_2 - 5x_3 &= -9 \\ x_3 &= 2 \end{aligned} \quad \text{or} \quad \begin{aligned} x_1 &= -x_2 - 2x_3 + 8 \\ x_2 &= 5x_3 - 9 \\ x_3 &= 2 \end{aligned}.$$

Since $x_3 = 2$, $x_2 = 5(2) - 9 = 1$, and $x_1 = -1 - 2(2) + 8 = 3$. The solution is $x_1 = 3$, $x_2 = 1$, $x_3 = 2$.

11. From Exercise 7, one row echelon form of the augmented matrix is $\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The corresponding system is

$$\begin{aligned} x - y + 2z - w &= -1 \\ y - 2z \qquad &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x &= y - 2z + w - 1 \\ y &= 2z \end{aligned}.$$

Let $z = s$ and $w = t$. Then $y = 2s$ and $x = 2s - 2s + t - 1 = t - 1$. The solution is $x = t - 1$, $y = 2s$, $z = s$, $w = t$.

13. Since the system has more unknowns (4) than equations (3), it has nontrivial solutions.

15. Since the system has more unknowns (3) than equations (2), it has nontrivial solutions.

17. The augmented matrix is $\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

Interchange the first and second rows, then add −2 times the new first row to the second row.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Multiply the second row by $-\frac{1}{3}$, then add −1 times the new second row to the third row.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Multiply the third row by $\frac{1}{2}$, then add the new third row to the second row.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Add −2 times the second row to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The solution, which can be read from the matrix, is $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

19. The augmented matrix is $\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix}$.

Multiply the first row by $\frac{1}{3}$, then add −5 times the new first row to the second row.

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} & 0 \end{bmatrix}$$

Multiply the second row by $-\frac{3}{8}$.

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}$$

Add $-\frac{1}{3}$ times the second row to the first row.

$$\begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}$$

This corresponds to the system

$$\begin{matrix} x_1 & +\frac{1}{4}x_3 & & = 0 \\ & x_2 + \frac{1}{4}x_3 & + x_4 & = 0 \end{matrix} \quad \text{or} \quad \begin{matrix} x_1 = -\frac{1}{4}x_3 \\ x_2 = -\frac{1}{4}x_3 - x_4 \end{matrix}.$$

Let $x_3 = 4s$ and $x_4 = t$. Then $x_1 = -s$ and $x_2 = -s - t$. The solution is $x_1 = -s$, $x_2 = -s - t$, $x_3 = 4s$, $x_4 = t$.

21. The augmented matrix is $\begin{bmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix}$.

Interchange the first and second rows.

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix}$$

Add −2 times the first row to the third row and 2 times the first row to the fourth row.

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix}$$

Multiply the second row by $\frac{1}{2}$, then add −3 times the new second row to the third row and −1 times the new second row to the fourth row.

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{bmatrix}$$

Add 10 times the third row to the fourth row, −2 times the third row to the second row, and 3 times the third row to the first row.

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system is

$$\begin{matrix} w & & -y & & = 0 \\ & x & +y & & = 0 \\ & & & z & = 0 \end{matrix} \quad \text{or} \quad \begin{matrix} w = y \\ x = -y \\ z = 0 \end{matrix}.$$

Let $y = t$. The solution is $w = t$, $x = -t$, $y = t$, $z = 0$.

23. The augmented matrix is $\begin{bmatrix} 2 & -1 & 3 & 4 & 9 \\ 1 & 0 & -2 & 7 & 11 \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{bmatrix}$.

Interchange the first and second rows.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 2 & -1 & 3 & 4 & 9 \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{bmatrix}$$

Add −2 times the first row to the second and fourth rows, and add −3 times the first row to the third row.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & -1 & 7 & -10 & -13 \\ 0 & -3 & 7 & -16 & -25 \\ 0 & 1 & 8 & -10 & -12 \end{bmatrix}$$

Multiply the second row by −1, then add 3 times the new second row to the third row and −1 times the new second row to the fourth row.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & -14 & 14 & 14 \\ 0 & 0 & 15 & -20 & -25 \end{bmatrix}$$

Multiply the third row by $-\frac{1}{14}$, then add −15 times the new third row to the fourth row.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -5 & -10 \end{bmatrix}$$

Multiply the fourth row by $-\frac{1}{5}$, then add the new fourth row to the third row, add −10 times the new fourth row to the second row, and add −7 times the new fourth row to the first row.

$$\begin{bmatrix} 1 & 0 & -2 & 0 & -3 \\ 0 & 1 & -7 & 0 & -7 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Add 7 times the third row to the second row and 2 times the third row to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution, which can be read from the matrix, is $I_1 = -1$, $I_2 = 0$, $I_3 = 1$, $I_4 = 2$.

25. The augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix}$$

Add −3 times the first row to the second row and −4 times the first row to the third row.

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$$

Add −1 times the second row to the third row.

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

The third row corresponds to the equation $(a^2-16)z = a-4$ or $(a+4)(a-4)z = a-4$.

If $a = -4$, this equation is $0z = -8$, which has no solution.

If $a = 4$, this equation is $0z = 0$, and z is a free variable. For any other value of a, the solution of this equation is $z = \frac{1}{a+4}$. Thus, if $a = 4$, the system has infinitely many solutions; if $a = -4$, the system has no solution; if $a \neq \pm 4$, the system has exactly one solution.

27. The augmented matrix is $\begin{bmatrix} 1 & 2 & 1 \\ 2 & a^2-5 & a-1 \end{bmatrix}$.

Add −2 times the first row to the second row.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & a^2-9 & a-3 \end{bmatrix}$$

The second row corresponds to the equation $(a^2-9)y = a-3$ or $(a+3)(a-3)y = a-3$.

If $a = -3$, this equation is $0y = -6$, which has no solution. If $a = 3$, this equation is $0y = 0$, and y is a free variable. For any other value of a, the solution of this equation is $y = \frac{1}{a+3}$.

Thus, if $a = -3$, the system has no solution; if $a = 3$, the system has infinitely many solutions; if $a \neq \pm 3$, the system has exactly one solution.

29. The augmented matrix is $\begin{bmatrix} 2 & 1 & a \\ 3 & 6 & b \end{bmatrix}$.

Multiply the first row by $\frac{1}{2}$, then add −3 times the new first row to the second row.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{a}{2} \\ 0 & \frac{9}{2} & -\frac{3a}{2}+b \end{bmatrix}$$

Multiply the second row by $\frac{2}{9}$, then add $-\frac{1}{2}$ times the new second row to the first row.

$$\begin{bmatrix} 1 & 0 & \frac{2a}{3}-\frac{b}{9} \\ 0 & 1 & -\frac{a}{3}+\frac{2b}{9} \end{bmatrix}$$

The solution is $x = \frac{2a}{3} - \frac{b}{9}$, $y = -\frac{a}{3} + \frac{2b}{9}$.

31. Add −2 times the first row to the second row.

$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ is in row echelon form. Add −3 times the second row to the first row.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is another row echelon form for the matrix.

33. Let $x = \sin \alpha$, $y = \cos \beta$, and $z = \tan \gamma$, then the system is

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + 5y + 3z &= 0 \\ -x - 5y + 5z &= 0 \end{aligned}$$

The augmented matrix is $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & -5 & 5 & 0 \end{bmatrix}$.

Add −2 times the first row to the second row, and add the first row to the third row.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & 8 & 0 \end{bmatrix}$$

Add 3 times the second row to the third row.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Multiply the third row by −1 then add 3 times the new third row to the second row and −3 times the new third row to the first row.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Add −2 times the second row to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The system has only the trivial solution, which corresponds to $\sin\alpha = 0$, $\cos\beta = 0$, $\tan\gamma = 0$.

For $0 \le \alpha \le 2\pi$, $\sin\alpha = 0 \Rightarrow \alpha = 0, \pi, 2\pi$.

For $0 \le \beta \le 2\pi$, $\cos\beta = 0 \Rightarrow \beta = \frac{\pi}{2}, \frac{3\pi}{2}$.

For $0 \le \gamma \le 2\pi$, $\tan\gamma = 0 \Rightarrow \gamma = 0, \pi, 2\pi$.

Thus, the original system has $3 \cdot 2 \cdot 3 = 18$ solutions.

35. Let $X = x^2$, $Y = y^2$, and $Z = z^2$, then the system is

$$\begin{aligned} X + Y + Z &= 6 \\ X - Y + 2Z &= 2 \\ 2X + Y - Z &= 3 \end{aligned}$$

The augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 2 \\ 2 & 1 & -1 & 3 \end{bmatrix}$.

Add −1 times the first row to the second row and −2 times the first row to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -4 \\ 0 & -1 & -3 & -9 \end{bmatrix}$$

Multiply the second row by $-\frac{1}{2}$, then add the new second row to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & -\frac{7}{2} & -7 \end{bmatrix}$$

Multiply the third row by $-\frac{2}{7}$, then add $\frac{1}{2}$ times the new third row to the second row and −1 times the new third row to the first row.

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Add −1 times the second row to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

This corresponds to the system $\begin{aligned} X \quad\quad &= 1 \\ Y \quad &= 3 \\ Z &= 2 \end{aligned}$.

$X = 1 \Rightarrow x = \pm 1$

$Y = 3 \Rightarrow y = \pm\sqrt{3}$

$Z = 2 \Rightarrow z = \pm\sqrt{2}$

The solutions are $x = \pm 1$, $y = \pm\sqrt{3}$, $z = \pm\sqrt{2}$.

37. (0, 10): $d = 10$

(1, 7): $a + b + c + d = 7$

(3, −11): $27a + 9b + 3c + d = -11$

(4, −14): $64a + 16b + 4c + d = -14$

The system is

$$\begin{aligned} a + b + c + d &= 7 \\ 27a + 9b + 3c + d &= -11 \\ 64a + 16b + 4c + d &= -14 \\ d &= 10 \end{aligned}$$

and the augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 27 & 9 & 3 & 1 & -11 \\ 64 & 16 & 4 & 1 & -14 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$.

Add −27 times the first row to the second row and −64 times the first row to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & -18 & -24 & -26 & -200 \\ 0 & -48 & -60 & -63 & -462 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$

Multiply the second row by $-\frac{1}{18}$, then add 48 times the new second row to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & \frac{4}{3} & \frac{13}{9} & \frac{100}{9} \\ 0 & 0 & 4 & \frac{19}{3} & \frac{214}{3} \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$

Multiply the third row by $\frac{1}{4}$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & \frac{4}{3} & \frac{13}{9} & \frac{100}{9} \\ 0 & 0 & 1 & \frac{19}{12} & \frac{107}{6} \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$

Add $-\frac{19}{12}$ times the fourth row to the third row, $-\frac{13}{9}$ times the fourth row to the second row, and −1 times the fourth row to the first row.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & -3 \\ 0 & 1 & \frac{4}{3} & 0 & -\frac{10}{3} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$

Add $-\dfrac{4}{3}$ times the third row to the second row and −1 times the third row to the first row.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$

Add −1 times the second row to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$

The coefficients are $a = 1$, $b = -6$, $c = 2$, $d = 10$.

39. Since the homogeneous system has only the trivial solution, using the same steps of Gauss-Jordan elimination will reduce the augmented matrix of the nonhomogeneous system to the form $\begin{bmatrix} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & d_3 \end{bmatrix}$.

Thus the system has exactly one solution.

41. **(a)** If $a \neq 0$, the sequence of matrices is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \to \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix} \to \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If $a = 0$, the sequence of matrices is

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \to \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \to \begin{bmatrix} 1 & \frac{d}{c} \\ 0 & b \end{bmatrix} \to \begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) Since $ad - bc \neq 0$, the lines are neither identical nor parallel, so the augmented matrix $\begin{bmatrix} a & b & k \\ c & d & l \end{bmatrix}$ will reduce to $\begin{bmatrix} 1 & 0 & k_1 \\ 0 & 1 & l_1 \end{bmatrix}$ and the system has exactly one solution.

43. **(a)** There are eight possibilities. p and q are any real numbers.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & p & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & p & q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) There are 16 possibilities. p, q, r, and s are any real numbers.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & p & 0 \\ 0 & 1 & q & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & p & q \\ 0 & 1 & r & s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & p & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & p & q & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 & p \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & p & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & p & q & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & p & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

True/False 1.2

(a) True; reduced row echelon form is a row echelon form.

(b) False; for instance, adding a nonzero multiple of the first row to any other row will result in a matrix that is not in row echelon form.

(c) False; see Exercise 31.

(d) True; the sum of the number of leading 1's and the number of free variables is n.

(e) True; a column cannot contain more than one leading 1.

(f) False; in row echelon form, there can be nonzero entries above the leading 1 in a column.

(g) True; since there are n leading 1's, there are no free variables, so only the trivial solution is possible.

(h) False; if the system has more equations than unknowns, a row of zeros does not indicate infinitely many solutions.

(i) False; the system could be inconsistent.

Section 1.3

Exercise Set 1.3

1. **(a)** BA is undefined.

(b) $AC + D$ is 4×2.

(c) $AE + B$ is undefined, since AE is 4×4 not 4×5.

(d) $AB + B$ is undefined since AB is undefined.

(e) $E(A + B)$ is 5×5.

(f) $E(AC)$ is 5×2.

(g) $E^T A$ is undefined since E^T is 4×5.

(h) $(A^T + E)D$ is 5×2.

3. **(a)** $$D+E = \begin{bmatrix} 1+6 & 5+1 & 2+3 \\ -1+(-1) & 0+1 & 1+2 \\ 3+4 & 2+1 & 4+3 \end{bmatrix} = \begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$$

(b) $$D-E = \begin{bmatrix} 1-6 & 5-1 & 2-3 \\ -1-(-1) & 0-1 & 1-2 \\ 3-4 & 2-1 & 4-3 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

(c) $$5A = \begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}$$

(d) $$-7C = \begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}$$

(e) $2B - C$ is not defined since $2B$ is a 2×2 matrix and C is a 2×3 matrix.

(f) $$4E-2D = \begin{bmatrix} 24 & 4 & 12 \\ -4 & 4 & 8 \\ 16 & 4 & 12 \end{bmatrix} - \begin{bmatrix} 2 & 10 & 4 \\ -2 & 0 & 2 \\ 6 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{bmatrix}$$

(g) $$\begin{aligned} -3(D+2E) &= -3\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 12 & 2 & 6 \\ -2 & 2 & 4 \\ 8 & 2 & 6 \end{bmatrix}\right) \\ &= -3\begin{bmatrix} 13 & 7 & 8 \\ -3 & 2 & 5 \\ 11 & 4 & 10 \end{bmatrix} \\ &= \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix} \end{aligned}$$

(h) $$A-A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(i) $\mathrm{tr}(D) = 1 + 0 + 4 = 5$

(j) $\operatorname{tr}(D-3E) = \operatorname{tr}\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 18 & 3 & 9 \\ -3 & 3 & 6 \\ 12 & 3 & 9 \end{bmatrix}\right)$

$= \operatorname{tr}\left(\begin{bmatrix} -17 & 2 & -7 \\ 2 & -3 & -5 \\ -9 & -1 & -5 \end{bmatrix}\right)$

$= -17 - 3 - 5$

$= -25$

(k) $4\operatorname{tr}(7B) = 4\operatorname{tr}\left(\begin{bmatrix} 28 & -7 \\ 0 & 14 \end{bmatrix}\right)$

$= 4(28+14)$

$= 4(42)$

$= 168$

(l) $\operatorname{tr}(A)$ is not defined because A is not a square matrix.

5. (a) $AB = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} (3\cdot 4)+(0\cdot 0) & -(3\cdot 1)+(0\cdot 2) \\ -(1\cdot 4)+(2\cdot 0) & (1\cdot 1)+(2\cdot 2) \\ (1\cdot 4)+(1\cdot 0) & -(1\cdot 1)+(1\cdot 2) \end{bmatrix}$

$= \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$

(b) BA is not defined since B is a 2×2 matrix and A is 3×2.

(c) $(3E)D = \begin{bmatrix} 18 & 3 & 9 \\ -3 & 3 & 6 \\ 12 & 3 & 9 \end{bmatrix}\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} (18\cdot 1)-(3\cdot 1)+(9\cdot 3) & (18\cdot 5)+(3\cdot 0)+(9\cdot 2) & (18\cdot 2)+(3\cdot 1)+(9\cdot 4) \\ -(3\cdot 1)-(3\cdot 1)+(6\cdot 3) & -(3\cdot 5)+(3\cdot 0)+(6\cdot 2) & -(3\cdot 2)+(3\cdot 1)+(6\cdot 4) \\ (12\cdot 1)-(3\cdot 1)+(9\cdot 3) & (12\cdot 5)+(3\cdot 0)+(9\cdot 2) & (12\cdot 2)+(3\cdot 1)+(9\cdot 4) \end{bmatrix}$

$= \begin{bmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{bmatrix}$

(d) $(AB)C = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$

$= \begin{bmatrix} (12\cdot 1)-(3\cdot 3) & (12\cdot 4)-(3\cdot 1) & (12\cdot 2)-(3\cdot 5) \\ -(4\cdot 1)+(5\cdot 3) & -(4\cdot 4)+(5\cdot 1) & -(4\cdot 2)+(5\cdot 5) \\ (4\cdot 1)+(1\cdot 3) & (4\cdot 4)+(1\cdot 1) & (4\cdot 2)+(1\cdot 5) \end{bmatrix}$

$= \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$

(e) $A(BC) = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\left(\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}\right)$

$$= \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (4\cdot 1)-(1\cdot 3) & (4\cdot 4)-(1\cdot 1) & (4\cdot 2)-(1\cdot 5) \\ (0\cdot 1)+(2\cdot 3) & (0\cdot 4)+(2\cdot 1) & (0\cdot 2)+(2\cdot 5) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} (3\cdot 1)+(0\cdot 6) & (3\cdot 15)+(0\cdot 2) & (3\cdot 3)+(0\cdot 10) \\ -(1\cdot 1)+(2\cdot 6) & -(1\cdot 15)+(2\cdot 2) & -(1\cdot 3)+(2\cdot 10) \\ (1\cdot 1)+(1\cdot 6) & (1\cdot 15)+(1\cdot 2) & (1\cdot 3)+(1\cdot 10) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$$

(f) $CC^T = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}$

$$= \begin{bmatrix} (1\cdot 1)+(4\cdot 4)+(2\cdot 2) & (1\cdot 3)+(4\cdot 1)+(2\cdot 5) \\ (3\cdot 1)+(1\cdot 4)+(5\cdot 2) & (3\cdot 3)+(1\cdot 1)+(5\cdot 5) \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix}$$

(g) $(DA)^T = \left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\right)^T$

$$= \left(\begin{bmatrix} (1\cdot 3)-(5\cdot 1)+(2\cdot 1) & (1\cdot 0)+(5\cdot 2)+(2\cdot 1) \\ -(1\cdot 3)-(0\cdot 1)+(1\cdot 1) & -(1\cdot 0)+(0\cdot 2)+(1\cdot 1) \\ (3\cdot 3)-(2\cdot 1)+(4\cdot 1) & (3\cdot 0)+(2\cdot 2)+(4\cdot 1) \end{bmatrix}\right)^T$$

$$= \left(\begin{bmatrix} 0 & 12 \\ -2 & 1 \\ 11 & 8 \end{bmatrix}\right)^T$$

$$= \begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$$

(h)
$$\begin{aligned}(C^T B)A^T &= \left(\begin{bmatrix}1 & 3\\ 4 & 1\\ 2 & 5\end{bmatrix}\begin{bmatrix}4 & -1\\ 0 & 2\end{bmatrix}\right)\begin{bmatrix}3 & -1 & 1\\ 0 & 2 & 1\end{bmatrix}\\
&= \begin{bmatrix}(1\cdot 4)+(3\cdot 0) & -(1\cdot 1)+(3\cdot 2)\\ (4\cdot 4)+(1\cdot 0) & -(4\cdot 1)+(1\cdot 2)\\ (2\cdot 4)+(5\cdot 0) & -(2\cdot 1)+(5\cdot 2)\end{bmatrix}\begin{bmatrix}3 & -1 & 1\\ 0 & 2 & 1\end{bmatrix}\\
&= \begin{bmatrix}4 & 5\\ 16 & -2\\ 8 & 8\end{bmatrix}\begin{bmatrix}3 & -1 & 1\\ 0 & 2 & 1\end{bmatrix}\\
&= \begin{bmatrix}(4\cdot 3)+(5\cdot 0) & -(4\cdot 1)+(5\cdot 2) & (4\cdot 1)+(5\cdot 1)\\ (16\cdot 3)-(2\cdot 0) & -(16\cdot 1)-(2\cdot 2) & (16\cdot 1)-(2\cdot 1)\\ (8\cdot 3)+(8\cdot 0) & -(8\cdot 1)+(8\cdot 2) & (8\cdot 1)+(8\cdot 1)\end{bmatrix}\\
&= \begin{bmatrix}12 & 6 & 9\\ 48 & -20 & 14\\ 24 & 8 & 16\end{bmatrix}\end{aligned}$$

(i)
$$\begin{aligned}\operatorname{tr}(DD^T) &= \operatorname{tr}\left(\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix}\begin{bmatrix}1 & -1 & 3\\ 5 & 0 & 2\\ 2 & 1 & 4\end{bmatrix}\right)\\
&= \operatorname{tr}\left(\begin{bmatrix}(1\cdot 1)+(5\cdot 5)+(2\cdot 2) & -(1\cdot 1)+(5\cdot 0)+(2\cdot 1) & (1\cdot 3)+(5\cdot 2)+(2\cdot 4)\\ -(1\cdot 1)+(0\cdot 5)+(1\cdot 2) & (1\cdot 1)+(0\cdot 0)+(1\cdot 1) & -(1\cdot 3)+(0\cdot 2)+(1\cdot 4)\\ (3\cdot 1)+(2\cdot 5)+(4\cdot 2) & -(3\cdot 1)+(2\cdot 0)+(4\cdot 1) & (3\cdot 3)+(2\cdot 2)+(4\cdot 4)\end{bmatrix}\right)\\
&= \operatorname{tr}\left(\begin{bmatrix}30 & 1 & 21\\ 1 & 2 & 1\\ 21 & 1 & 29\end{bmatrix}\right)\\
&= 30+2+29\\
&= 61\end{aligned}$$

(j)
$$\begin{aligned}\operatorname{tr}(4E^T - D) &= \operatorname{tr}\left(\begin{bmatrix}24 & -4 & 16\\ 4 & 4 & 4\\ 12 & 8 & 12\end{bmatrix}-\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix}\right)\\
&= \operatorname{tr}\left(\begin{bmatrix}23 & -9 & 14\\ 5 & 4 & 3\\ 9 & 6 & 8\end{bmatrix}\right)\\
&= 23+4+8\\
&= 35\end{aligned}$$

(k) $\operatorname{tr}(C^T A^T + 2E^T) = \operatorname{tr}\left(\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix}\right)$

$$= \operatorname{tr}\left(\begin{bmatrix} (1\cdot 3)+(3\cdot 0) & -(1\cdot 1)+(3\cdot 2) & (1\cdot 1)+(3\cdot 1) \\ (4\cdot 3)+(1\cdot 0) & -(4\cdot 1)+(1\cdot 2) & (4\cdot 1)+(1\cdot 1) \\ (2\cdot 3)+(5\cdot 0) & -(2\cdot 1)+(5\cdot 2) & (2\cdot 1)+(5\cdot 1) \end{bmatrix} + \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix}\right)$$

$$= \operatorname{tr}\left(\begin{bmatrix} 3 & 5 & 4 \\ 12 & -2 & 5 \\ 6 & 8 & 7 \end{bmatrix} + \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix}\right)$$

$$= \operatorname{tr}\left(\begin{bmatrix} 15 & 3 & 12 \\ 14 & 0 & 7 \\ 12 & 12 & 13 \end{bmatrix}\right)$$

$$= 15 + 0 + 13$$

$$= 28$$

(l) $\operatorname{tr}((EC^T)^T A) = \operatorname{tr}\left(\left(\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}\right)^T \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\right)$

$$= \operatorname{tr}\left(\left(\begin{bmatrix} (6\cdot 1)+(1\cdot 4)+(3\cdot 2) & (6\cdot 3)+(1\cdot 1)+(3\cdot 5) \\ -(1\cdot 1)+(1\cdot 4)+(2\cdot 2) & -(1\cdot 3)+(1\cdot 1)+(2\cdot 5) \\ (4\cdot 1)+(1\cdot 4)+(3\cdot 2) & (4\cdot 3)+(1\cdot 1)+(3\cdot 5) \end{bmatrix}\right)^T \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\right)$$

$$= \operatorname{tr}\left(\left(\begin{bmatrix} 16 & 34 \\ 7 & 8 \\ 14 & 28 \end{bmatrix}^T\right)\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\right)$$

$$= \operatorname{tr}\left(\begin{bmatrix} 16 & 7 & 14 \\ 34 & 8 & 28 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\right)$$

$$= \operatorname{tr}\left(\begin{bmatrix} (16\cdot 3)-(7\cdot 1)+(14\cdot 1) & (16\cdot 0)+(7\cdot 2)+(14\cdot 1) \\ (34\cdot 3)-(8\cdot 1)+(28\cdot 1) & (34\cdot 0)+(8\cdot 2)+(28\cdot 1) \end{bmatrix}\right)$$

$$= \operatorname{tr}\left(\begin{bmatrix} 55 & 28 \\ 122 & 44 \end{bmatrix}\right)$$

$$= 55 + 44$$

$$= 99$$

7. (a) The first row of AB is the first row of A times B.

$$[\mathbf{a}_1 \quad B] = [3 \quad -2 \quad 7]\begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

$$= [18+0+49 \quad -6-2+49 \quad 12-6+35]$$

$$= [67 \quad 41 \quad 41]$$

(b) The third row of AB is the third row of A times B.

$$\begin{aligned}[\mathbf{a}_3 \quad B] &= [0 \quad 4 \quad 9]\begin{bmatrix}6 & -2 & 4\\ 0 & 1 & 3\\ 7 & 7 & 5\end{bmatrix}\\ &= [0+0+63 \quad 0+4+63 \quad 0+12+45]\\ &= [63 \quad 67 \quad 57]\end{aligned}$$

(c) The second column of AB is A times the second column of B.

$$\begin{aligned}[A \quad \mathbf{b}_2] &= \begin{bmatrix}3 & -2 & 7\\ 6 & 5 & 4\\ 0 & 4 & 9\end{bmatrix}\begin{bmatrix}-2\\ 1\\ 7\end{bmatrix}\\ &= \begin{bmatrix}-6-2+49\\ -12+5+28\\ 0+4+63\end{bmatrix}\\ &= \begin{bmatrix}41\\ 21\\ 67\end{bmatrix}\end{aligned}$$

(d) The first column of BA is B times the first column of A.

$$\begin{aligned}[B \quad \mathbf{a}_1] &= \begin{bmatrix}6 & -2 & 4\\ 0 & 1 & 3\\ 7 & 7 & 5\end{bmatrix}\begin{bmatrix}3\\ 6\\ 0\end{bmatrix}\\ &= \begin{bmatrix}18-12+0\\ 0+6+0\\ 21+42+0\end{bmatrix}\\ &= \begin{bmatrix}6\\ 6\\ 63\end{bmatrix}\end{aligned}$$

(e) The third row of AA is the third row of A times A.

$$\begin{aligned}[\mathbf{a}_3 \quad A] &= [0 \quad 4 \quad 9]\begin{bmatrix}3 & -2 & 7\\ 6 & 5 & 4\\ 0 & 4 & 9\end{bmatrix}\\ &= [0+24+0 \quad 0+20+36 \quad 0+16+81]\\ &= [24 \quad 56 \quad 97]\end{aligned}$$

(f) The third column of AA is A times the third column of A.

$$\begin{aligned}[A \quad \mathbf{a}_3] &= \begin{bmatrix}3 & -2 & 7\\ 6 & 5 & 4\\ 0 & 4 & 9\end{bmatrix}\begin{bmatrix}7\\ 4\\ 9\end{bmatrix}\\ &= \begin{bmatrix}21-8+63\\ 42+20+36\\ 0+16+81\end{bmatrix}\\ &= \begin{bmatrix}76\\ 98\\ 97\end{bmatrix}\end{aligned}$$

9. (a) $AA = \begin{bmatrix}-3 & 12 & 76\\ 48 & 29 & 98\\ 24 & 56 & 97\end{bmatrix}$

$$\begin{bmatrix}-3\\ 48\\ 24\end{bmatrix} = 3\begin{bmatrix}3\\ 6\\ 0\end{bmatrix} + 6\begin{bmatrix}-2\\ 5\\ 4\end{bmatrix}$$

$$\begin{bmatrix}12\\ 29\\ 56\end{bmatrix} = -2\begin{bmatrix}3\\ 6\\ 0\end{bmatrix} + 5\begin{bmatrix}-2\\ 5\\ 4\end{bmatrix} + 4\begin{bmatrix}7\\ 4\\ 9\end{bmatrix}$$

$$\begin{bmatrix}76\\ 98\\ 97\end{bmatrix} = 7\begin{bmatrix}3\\ 6\\ 0\end{bmatrix} + 4\begin{bmatrix}-2\\ 5\\ 4\end{bmatrix} + 9\begin{bmatrix}7\\ 4\\ 9\end{bmatrix}$$

(b) $BB = \begin{bmatrix}64 & 14 & 38\\ 21 & 22 & 18\\ 77 & 28 & 74\end{bmatrix}$

$$\begin{bmatrix}64\\ 21\\ 77\end{bmatrix} = 6\begin{bmatrix}6\\ 0\\ 7\end{bmatrix} + 7\begin{bmatrix}4\\ 3\\ 5\end{bmatrix}$$

$$\begin{bmatrix}14\\ 22\\ 28\end{bmatrix} = -2\begin{bmatrix}6\\ 0\\ 7\end{bmatrix} + 1\begin{bmatrix}-2\\ 1\\ 7\end{bmatrix} + 7\begin{bmatrix}4\\ 3\\ 5\end{bmatrix}$$

$$\begin{bmatrix}38\\ 18\\ 74\end{bmatrix} = 4\begin{bmatrix}6\\ 0\\ 7\end{bmatrix} + 3\begin{bmatrix}-2\\ 1\\ 7\end{bmatrix} + 5\begin{bmatrix}4\\ 3\\ 5\end{bmatrix}$$

11. (a) $A = \begin{bmatrix}2 & -3 & 5\\ 9 & -1 & 1\\ 1 & 5 & 4\end{bmatrix}$, $\mathbf{x} = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}7\\ -1\\ 0\end{bmatrix}$

The equation is $\begin{bmatrix}2 & -3 & 5\\ 9 & -1 & 1\\ 1 & 5 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}7\\ -1\\ 0\end{bmatrix}$.

(b) $A = \begin{bmatrix} 4 & 0 & -3 & 1 \\ 5 & 1 & 0 & -8 \\ 2 & -5 & 9 & -1 \\ 0 & 3 & -1 & 7 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$

The equation is

$$\begin{bmatrix} 4 & 0 & -3 & 1 \\ 5 & 1 & 0 & -8 \\ 2 & -5 & 9 & -1 \\ 0 & 3 & -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}.$$

13. (a) The system is

$$\begin{aligned} 5x_1 + 6x_2 - 7x_3 &= 2 \\ -x_1 - 2x_2 + 3x_3 &= 0 \\ 4x_2 - x_3 &= 3 \end{aligned}$$

(b) The system is

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 &= 2 \\ 5x_1 - 3x_2 - 6x_3 &= -9 \end{aligned}$$

15. $[k \quad 1 \quad 1]\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix}$

$= [k+1 \quad k+2 \quad -1]\begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix}$

$= [k^2 + k + k + 2 - 1]$

$= [k^2 + 2k + 1]$

The only solution of $k^2 + 2k + 1 = 0$ is $k = -1$.

17. $\begin{bmatrix} a & 3 \\ -1 & a+b \end{bmatrix} = \begin{bmatrix} 4 & d-2c \\ d+2c & -2 \end{bmatrix}$ gives the equations

$a = 4$
$d - 2c = 3$
$d + 2c = -1$
$a + b = -2$
$a = 4 \Rightarrow b = -6$
$d - 2c = 3$, $d + 2c = -1 \Rightarrow 2d = 2$, $d = 1$
$d = 1 \Rightarrow 2c = -2$, $c = -1$
The solution is $a = 4$, $b = -6$, $c = -1$, $d = 1$.

19. The *ij*-entry of *kA* is ka_{ij}. If $ka_{ij} = 0$ for all *i*, *j* then either $k = 0$ or $a_{ij} = 0$ for all *i*, *j*, in which case *A* = *0*.

21. The diagonal entries of *A* + *B* are $a_{ii} + b_{ii}$, so

$$\begin{aligned} &\operatorname{tr}(A+B) \\ &= (a_{11} + b_{11}) + (a_{22} + b_{22}) + \cdots + (a_{nn} + b_{nn}) \\ &= (a_{11} + a_{22} + \cdots + a_{nn}) + (b_{11} + b_{22} + \cdots + b_{nn}) \\ &= \operatorname{tr}(A) + \operatorname{tr}(B) \end{aligned}$$

23. (a) $\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix}$

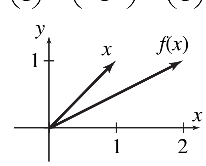
(b) $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ 0 & 0 & a_{33} & a_{34} & a_{35} & a_{36} \\ 0 & 0 & 0 & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix}$

(c) $\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$

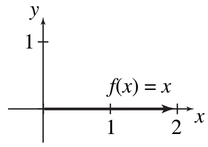
(d) $\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{bmatrix}$

25. $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix}$

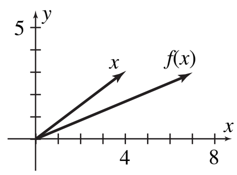
(a) $f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



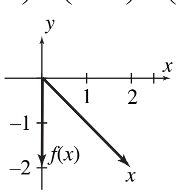
(b) $f\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}2+0\\0\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$



(c) $f\begin{pmatrix}4\\3\end{pmatrix}=\begin{pmatrix}4+3\\3\end{pmatrix}=\begin{pmatrix}7\\3\end{pmatrix}$



(d) $f\begin{pmatrix}2\\-2\end{pmatrix}=\begin{pmatrix}2-2\\-2\end{pmatrix}=\begin{pmatrix}0\\-2\end{pmatrix}$



27. Let $A=[a_{ij}]$.

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}a_{11}x+a_{12}y+a_{13}z\\ a_{21}x+a_{22}y+a_{23}z\\ a_{31}x+a_{32}y+a_{33}z\end{bmatrix}=\begin{bmatrix}x+y\\x-y\\0\end{bmatrix}$$

The matrix equation yields the equations

$$\begin{aligned}a_{11}x+a_{12}y+a_{13}z&=x+y\\ a_{21}x+a_{22}y+a_{23}z&=x-y\ .\\ a_{31}x+a_{32}y+a_{33}z&=0\end{aligned}$$

For these equations to be true for all values of x, y, and z it must be that $a_{11}=1$, $a_{12}=1$, $a_{21}=1$, $a_{22}=-1$, and $a_{ij}=0$ for all other i, j. There is one such matrix, $A=\begin{bmatrix}1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$.

29. **(a)** Both $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ and $\begin{bmatrix}-1 & -1\\-1 & -1\end{bmatrix}$ are square roots of $\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$.

(b) The four square roots of $\begin{bmatrix}5 & 0\\0 & 9\end{bmatrix}$ are $\begin{bmatrix}\sqrt{5} & 0\\0 & 3\end{bmatrix}$, $\begin{bmatrix}-\sqrt{5} & 0\\0 & 3\end{bmatrix}$, $\begin{bmatrix}\sqrt{5} & 0\\0 & -3\end{bmatrix}$, and $\begin{bmatrix}-\sqrt{5} & 0\\0 & -3\end{bmatrix}$.

True/False 1.3

(a) True; only square matrices have main diagonals.

(b) False; an $m\times n$ matrix has m row vectors and n column vectors.

(c) False; matrix multiplication is not commutative.

(d) False; the ith row vector of AB is found by multiplying the ith row vector of A by B.

(e) True

(f) False; for example, if $A=\begin{bmatrix}1 & 3\\4 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$, then $AB=\begin{bmatrix}1 & 6\\4 & -2\end{bmatrix}$ and tr(AB) = −1, while tr(A) = 0 and tr(B) = 3.

(g) False; for example, if $A=\begin{bmatrix}1 & 3\\4 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$, then $(AB)^T=\begin{bmatrix}1 & 4\\6 & -2\end{bmatrix}$, while $A^TB^T=\begin{bmatrix}1 & 8\\3 & -2\end{bmatrix}$.

(h) True; for a square matrix A the main diagonals of A and A^T are the same.

(i) True; if A is a 6×4 matrix and B is an $m\times n$ matrix, then A^T is a 4×6 matrix and B^T is an $n\times m$ matrix. So B^TA^T is only defined if $m = 4$, and it can only be a 2×6 matrix if $n = 2$.

(j) True,
$$\begin{aligned}\operatorname{tr}(cA)&=ca_{11}+ca_{22}+\cdots+ca_{nn}\\ &=c(a_{11}+a_{22}+\cdots+a_{nn})\\ &=c\ \operatorname{tr}(A)\end{aligned}$$

(k) True; if $A - C = B - C$, then $a_{ij}-c_{ij}=b_{ij}-c_{ij}$, so it follows that $a_{ij}=b_{ij}$.

(l) False; for example, if $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}1 & 5\\ 3 & 7\end{bmatrix}$, and $C=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, then $AC=BC=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$.

(m) True; if A is an $m\times n$ matrix, then for both AB and BA to be defined B must be an $n\times m$ matrix. Then AB will be an $m\times m$ matrix and BA will be an $n\times n$ matrix, so $m=n$ in order for the sum to be defined.

(n) True; since the jth column vector of AB is A[jth column vector of B], a column of zeros in B will result in a column of zeros in AB.

(o) False; $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 5\\ 3 & 7\end{bmatrix}=\begin{bmatrix}1 & 5\\ 0 & 0\end{bmatrix}$

Section 1.4

Exercise Set 1.4

5.
$$\begin{aligned}B^{-1}&=\frac{1}{2\cdot 4-(-3)4}\begin{bmatrix}4 & 3\\ -4 & 2\end{bmatrix}\\ &=\frac{1}{20}\begin{bmatrix}4 & 3\\ -4 & 2\end{bmatrix}\\ &=\begin{bmatrix}\frac{1}{5} & \frac{3}{20}\\ -\frac{1}{5} & \frac{1}{10}\end{bmatrix}\end{aligned}$$

7. $D^{-1}=\dfrac{1}{2\cdot 3-0}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}=\dfrac{1}{6}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{bmatrix}$

9. Here, $a=d=\dfrac{1}{2}(e^x+e^{-x})$ and $b=c=\dfrac{1}{2}(e^x-e^{-x})$, so

$$\begin{aligned}ad-bc&=\frac{1}{4}(e^x+e^{-x})^2-\frac{1}{4}(e^x-e^{-x})^2\\ &=\frac{1}{4}(e^{2x}+2+e^{-2x})-\frac{1}{4}(e^{2x}-2+e^{-2x})\\ &=1.\end{aligned}$$

Thus,
$$\begin{bmatrix}\frac{1}{2}(e^x+e^{-x}) & \frac{1}{2}(e^x-e^{-x})\\ \frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x+e^{-x})\end{bmatrix}^{-1}=\begin{bmatrix}\frac{1}{2}(e^x+e^{-x}) & -\frac{1}{2}(e^x-e^{-x})\\ -\frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x+e^{-x})\end{bmatrix}.$$

11. $B^T=\begin{bmatrix}2 & 4\\ -3 & 4\end{bmatrix}$;

$$(B^T)^{-1}=\frac{1}{8+12}\begin{bmatrix}4 & -4\\ 3 & 2\end{bmatrix}=\begin{bmatrix}\frac{1}{5} & -\frac{1}{5}\\ \frac{3}{20} & \frac{1}{10}\end{bmatrix}=(B^{-1})^T$$

13. $ABC=\begin{bmatrix}70 & 45\\ 122 & 79\end{bmatrix}$

$$\begin{aligned}(ABC)^{-1}&=\frac{1}{70\cdot 79-122\cdot 45}\begin{bmatrix}79 & -45\\ -122 & 70\end{bmatrix}\\ &=\frac{1}{40}\begin{bmatrix}79 & -45\\ -122 & 70\end{bmatrix}\end{aligned}$$

$$C^{-1}=\frac{1}{6(-1)-4(-2)}\begin{bmatrix}-1 & -4\\ 2 & 6\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1 & -4\\ 2 & 6\end{bmatrix}$$

$$B^{-1}=\frac{1}{2\cdot 4-(-3)4}\begin{bmatrix}4 & 3\\ -4 & 2\end{bmatrix}=\frac{1}{20}\begin{bmatrix}4 & 3\\ -4 & 2\end{bmatrix}$$

$$A^{-1}=\frac{1}{3\cdot 2-1\cdot 5}\begin{bmatrix}2 & -1\\ -5 & 3\end{bmatrix}=\begin{bmatrix}2 & -1\\ -5 & 3\end{bmatrix}$$

$$\begin{aligned}C^{-1}B^{-1}A^{-1}&=\frac{1}{40}\left(\begin{bmatrix}-1 & -4\\ 2 & 6\end{bmatrix}\begin{bmatrix}4 & 3\\ -4 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\ -5 & 3\end{bmatrix}\right)\\ &=\frac{1}{40}\begin{bmatrix}79 & -45\\ -122 & 70\end{bmatrix}\end{aligned}$$

15.
$$\begin{aligned}7A&=((7A)^{-1})^{-1}\\ &=\frac{1}{(-3)(-2)-1\cdot 7}\begin{bmatrix}-2 & -7\\ -1 & -3\end{bmatrix}\\ &=-1\begin{bmatrix}-2 & -7\\ -1 & -3\end{bmatrix}\\ &=\begin{bmatrix}2 & 7\\ 1 & 3\end{bmatrix}\end{aligned}$$

Thus $A=\begin{bmatrix}\frac{2}{7} & 1\\ \frac{1}{7} & \frac{3}{7}\end{bmatrix}$.

17.
$$\begin{aligned}I+2A&=((I+2A)^{-1})^{-1}\\ &=\frac{1}{-1\cdot 5-2\cdot 4}\begin{bmatrix}5 & -2\\ -4 & -1\end{bmatrix}\\ &=-\frac{1}{13}\begin{bmatrix}5 & -2\\ -4 & -1\end{bmatrix}\\ &=\begin{bmatrix}-\frac{5}{13} & \frac{2}{13}\\ \frac{4}{13} & \frac{1}{13}\end{bmatrix}\end{aligned}$$

Thus $2A = \begin{bmatrix} -\frac{5}{13} & \frac{2}{13} \\ \frac{4}{13} & \frac{1}{13} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{18}{13} & \frac{2}{13} \\ \frac{4}{13} & -\frac{12}{13} \end{bmatrix}$ and $A = \begin{bmatrix} -\frac{9}{13} & \frac{1}{13} \\ \frac{2}{13} & -\frac{6}{13} \end{bmatrix}$.

19. (a) $$\begin{aligned} A^3 &= \left(\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\right)\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 41 & 15 \\ 30 & 11 \end{bmatrix} \end{aligned}$$

(b) $$\begin{aligned} A^{-3} &= (A^3)^{-1} \\ &= \frac{1}{41\cdot 11 - 15\cdot 30}\begin{bmatrix} 11 & -15 \\ -30 & 41 \end{bmatrix} \\ &= \begin{bmatrix} 11 & -15 \\ -30 & 41 \end{bmatrix} \end{aligned}$$

(c) $$\begin{aligned} A^2 - 2A + I &= \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix} \end{aligned}$$

(d) $$p(A) = A - 2I = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

(e) $$\begin{aligned} p(A) &= 2A^2 - A + I \\ &= \begin{bmatrix} 22 & 8 \\ 16 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 20 & 7 \\ 14 & 6 \end{bmatrix} \end{aligned}$$

(f) $$\begin{aligned} p(A) &= A^3 - 2A + 4I \\ &= \begin{bmatrix} 41 & 15 \\ 30 & 11 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 39 & 13 \\ 26 & 13 \end{bmatrix} \end{aligned}$$

21. (a) $$\begin{aligned} A^3 &= \left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -3 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -3 & -1 \end{bmatrix}\right)\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 0 & 0 \\ 0 & -8 & -6 \\ 0 & 6 & -8 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 27 & 0 & 0 \\ 0 & 26 & -18 \\ 0 & 18 & 26 \end{bmatrix} \end{aligned}$$

(b) Note that the inverse of $\begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$ is $\frac{1}{10}\begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} & -\frac{3}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix}$. Thus $A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{10} & -\frac{3}{10} \\ 0 & \frac{3}{10} & -\frac{1}{10} \end{bmatrix}$.

$$\begin{aligned} A^{-3} = (A^{-1})^3 &= \left(\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{10} & -\frac{3}{10} \\ 0 & \frac{3}{10} & -\frac{1}{10} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{10} & -\frac{3}{10} \\ 0 & \frac{3}{10} & -\frac{1}{10} \end{bmatrix}\right)\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{10} & -\frac{3}{10} \\ 0 & \frac{3}{10} & -\frac{1}{10} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{9} & 0 & 0 \\ 0 & -\frac{2}{25} & \frac{3}{50} \\ 0 & -\frac{3}{50} & -\frac{2}{25} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{10} & -\frac{3}{10} \\ 0 & \frac{3}{10} & -\frac{1}{10} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{27} & 0 & 0 \\ 0 & 0.026 & 0.018 \\ 0 & -0.018 & 0.026 \end{bmatrix} \end{aligned}$$

(c) $$\begin{aligned} A^2 - 2A + I &= \begin{bmatrix} 9 & 0 & 0 \\ 0 & -8 & -6 \\ 0 & 6 & -8 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 6 \\ 0 & -6 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -5 & -12 \\ 0 & 12 & -5 \end{bmatrix} \end{aligned}$$

(d) $$\begin{aligned} p(A) = A - 2I &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -3 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & -3 & -3 \end{bmatrix} \end{aligned}$$

(e) $$\begin{aligned} p(A) &= 2A^2 - A + I \\ &= \begin{bmatrix} 18 & 0 & 0 \\ 0 & -16 & -12 \\ 0 & 12 & -16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -3 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 16 & 0 & 0 \\ 0 & -14 & -15 \\ 0 & 15 & -14 \end{bmatrix} \end{aligned}$$

(f) $$\begin{aligned} p(A) &= A^3 - 2A + 4I \\ &= \begin{bmatrix} 27 & 0 & 0 \\ 0 & 26 & -18 \\ 0 & 18 & 26 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 6 \\ 0 & -6 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 25 & 0 & 0 \\ 0 & 32 & -24 \\ 0 & 24 & 32 \end{bmatrix} \end{aligned}$$

27. Since $a_{11}a_{22}\cdots a_{nn} \neq 0,$ none of the diagonal entries are zero.

Let $B = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}.$

$AB = BA = I$ so A is invertible and $B = A^{-1}.$

29. **(a)** If A has a row of zeros, then by formula (9) of Section 1.3, AB will have a row of zeros, so A cannot be invertible.

(b) If B has a column of zeros, then by formula (8) of Section 1.3, AB will have a column of zeros, and B cannot be invertible.

31.
$$\begin{aligned} C^TB^{-1}A^2BAC^{-1}DA^{-2}B^TC^{-2} &= C^T \\ CA^{-1}B^{-1}A^{-2}B(C^T)^{-1}\cdot C^TB^{-1}A^2BAC^{-1}DA^{-2}B^TC^{-2} &= CA^{-1}B^{-1}A^{-2}B(C^T)^{-1}C^T \\ DA^{-2}B^TC^{-2} &= CA^{-1}B^{-1}A^{-2}B \\ DA^{-2}B^TC^{-2}\cdot C^2(B^T)^{-1}A^2 &= CA^{-1}B^{-1}A^{-2}B\cdot C^2(B^T)^{-1}A^2 \\ D &= CA^{-1}B^{-1}A^{-2}BC^2(B^T)^{-1}A^2 \end{aligned}$$

33.
$$\begin{aligned} (AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1} &= (B^{-1}A^{-1})(AC^{-1})(CD)D^{-1} \\ &= B^{-1}(A^{-1}A)(C^{-1}C)(DD^{-1}) \\ &= B^{-1} \end{aligned}$$

35. Let $X = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$

$$AX = I$$
$$\begin{bmatrix} b_{11}+b_{31} & b_{12}+b_{32} & b_{13}+b_{33} \\ b_{11}+b_{21} & b_{12}+b_{22} & b_{13}+b_{23} \\ b_{21}+b_{31} & b_{22}+b_{32} & b_{23}+b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consider the equations from the first columns of the matrices.

$b_{11}+b_{31} = 1$
$b_{11}+b_{21} = 0$
$b_{21}+b_{31} = 0$

Thus $b_{11} = b_{31} = -b_{21},$ so $b_{11} = \frac{1}{2},$ $b_{21} = -\frac{1}{2},$ and $b_{31} = \frac{1}{2}.$

Working similarly with the other columns of both matrices yields $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$

37. Let $X = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$

$$AX = I$$

$$\begin{bmatrix} b_{31} & b_{32} & b_{33} \\ b_{11}+b_{21} & b_{12}+b_{22} & b_{13}+b_{23} \\ -b_{11}+b_{21}+b_{31} & -b_{12}+b_{22}+b_{32} & -b_{13}+b_{23}+b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consider the equations from the first columns of the matrices.

$$\begin{aligned} b_{31} &= 1 \\ b_{11}+b_{21} &= 0 \\ -b_{11}+b_{21}+b_{31} &= 0 \end{aligned}$$

Thus, $b_{21} = -b_{11}$ and $-2b_{11}+1=0,$ so $b_{11} = \frac{1}{2},$ $b_{21} = -\frac{1}{2},$ and $b_{31} = 1.$

Working similarly with the other columns of both matrices yields

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}.$$

39. The matrix form of the system is $\begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$ so the solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ 3 \end{bmatrix}.$

$$\begin{aligned} \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}^{-1} &= \frac{1}{3\cdot 5-(-2)4}\begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix} \\ &= \frac{1}{23}\begin{bmatrix} 5 & 2 \\ -4 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{23}\begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{23}\begin{bmatrix} -5+6 \\ 4+9 \end{bmatrix} = \frac{1}{23}\begin{bmatrix} 1 \\ 13 \end{bmatrix}$$

The solution is $x_1 = \frac{1}{23},$ $x_2 = \frac{13}{23}.$

41. The matrix form of the system is $\begin{bmatrix} 6 & 1 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix},$ so the solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & -3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -2 \end{bmatrix}.$

$$\begin{aligned} \begin{bmatrix} 6 & 1 \\ 4 & -3 \end{bmatrix}^{-1} &= \frac{1}{6(-3)-1\cdot 4}\begin{bmatrix} -3 & -1 \\ -4 & 6 \end{bmatrix} \\ &= -\frac{1}{22}\begin{bmatrix} -3 & -1 \\ -4 & 6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= -\frac{1}{22}\begin{bmatrix} -3 & -1 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} \\ &= -\frac{1}{22}\begin{bmatrix} 0+2 \\ 0-12 \end{bmatrix} \\ &= -\frac{1}{22}\begin{bmatrix} 2 \\ -12 \end{bmatrix} \end{aligned}$$

The solution is $x_1 = -\frac{1}{11},$ $x_2 = \frac{6}{11}.$

51. (a) If A is invertible, then A^{-1} exists.

$$\begin{aligned} AB &= AC \\ A^{-1}AB &= A^{-1}AC \\ IB &= IC \\ B &= C \end{aligned}$$

(b) The matrix A in Example 3 is not invertible.

53. (a)
$$\begin{aligned} A(A^{-1}+B^{-1})B(A+B)^{-1} &= (AA^{-1}+AB^{-1})B(A+B)^{-1} \\ &= (I+AB^{-1})B(A+B)^{-1} \\ &= (IB+AB^{-1}B)(A+B)^{-1} \\ &= (B+A)(A+B)^{-1} \\ &= (A+B)(A+B)^{-1} \\ &= I \end{aligned}$$

(b) $A^{-1}+B^{-1} \neq (A+B)^{-1}$

55.
$$\begin{aligned} (I-A)(I+A+A^2+\cdots+A^{k-1}) &= (I-A)I+(I-A)A+(I-A)A^2+\cdots+(I-A)A^{k-1} \\ &= I-A+A-A^2+A^2-A^3+\cdots+A^{k-1}-A^k \\ &= I-A^k \\ &= I-0 \\ &= I \end{aligned}$$

True/False 1.4

(a) False; A and B are inverses if and only if $AB = BA = I$.

(b) False; $(A+B)^2 = (A+B)(A+B)$
$= A^2+AB+BA+B^2$
Since $AB \neq BA$ the two terms cannot be combined.

(c) False; $(A-B)(A+B) = A^2+AB-BA-B^2$.
Since $AB \neq BA$, the two terms cannot be combined.

(d) False; AB is invertible, but $(AB)^{-1} = B^{-1}A^{-1} \neq A^{-1}B^{-1}$.

(e) False; if A is an $m \times n$ matrix and B is and $n \times p$ matrix, with $m \neq p$, then AB and $(AB)^T$ are defined but $A^T B^T$ is not defined.

(f) True; by Theorem 1.4.5.

(g) True; $(kA+B)^T = (kA)^T + B^T = kA^T + B^T$.

(h) True; by Theorem 1.4.9.

(i) False; $p(I) = (a_0+a_1+a_2+\cdots+a_m)I$ which is a matrix, not a scalar.

(j) True, if the square matrix A has a row of zeros, the product of A and any matrix B will also have a row or column of zeros (depending on whether the product is AB or BA), so $AB = I$ is impossible. A similar statement can be made if A has a column of zeros.

(k) False; for example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are both invertible, but their sum $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

Section 1.5

Exercise Set 1.5

1. (a) The matrix results from adding −5 times the first row of I_2 to the second row; it is an elementary matrix.

(b) The matrix is not elementary; more than one elementary row operation on I_2 is required.

(c) The matrix is not elementary; more than one elementary row operation on I_3 is required.

(d) The matrix is not elementary; more than one elementary row operation on I_4 is required.

3. (a) The operation is to add 3 times the second row to the first row. The matrix is $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

(b) The operation is to multiply the first row by $-\frac{1}{7}$. The matrix is $\begin{bmatrix} -\frac{1}{7} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) The operation is to add 5 times the first row to the third row. The matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$.

(d) The operation is to interchange the first and third rows. The matrix is $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

5. (a) E interchanges the first and second rows of A.

$$EA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -6 & -6 & -6 \\ -1 & -2 & 5 & -1 \end{bmatrix}$$

(b) E adds −3 times the second row to the third row.

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ -1 & 9 & 4 & -12 & -10 \end{bmatrix}$$

(c) E adds 4 times the third row to the first row.

$$EA = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 13 & 28 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

7. (a) To obtain B from A, the first and third rows must be interchanged.

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) To obtain A from B, the first and third rows must be interchanged.

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) To obtain C from A, add −2 times the first row to the third row.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

(d) To obtain A from C, add 2 times the first row to the third row.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

9. $\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$

Add −2 times the first row to the second.

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

Multiply the second row by −1.

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

Add −4 times the second row to the first.

$$\left[\begin{array}{cc|cc} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

11. $\left[\begin{array}{cc|cc} -1 & 3 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{array}\right]$

Multiply the first row by −1.

$$\left[\begin{array}{cc|cc} 1 & -3 & -1 & 0 \\ 3 & -2 & 0 & 1 \end{array}\right]$$

Add −3 times the first row to the second.

$$\left[\begin{array}{cc|cc} 1 & -3 & -1 & 0 \\ 0 & 7 & 3 & 1 \end{array}\right]$$

Multiply the second row by $\frac{1}{7}$.

$$\left[\begin{array}{cc|cc} 1 & -3 & -1 & 0 \\ 0 & 1 & \frac{3}{7} & \frac{1}{7} \end{array}\right]$$

Add 3 times the second row to the first.

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{7} & \frac{3}{7} \\ 0 & 1 & \frac{3}{7} & \frac{1}{7} \end{array}\right]$$

$$\begin{bmatrix} -1 & 3 \\ 3 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{1}{7} \end{bmatrix}$$

13. $\left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$

Interchange the first and second rows.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$

Add −3 times the first row to the second and −2 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$

Interchange the second and third rows.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \\ 0 & 4 & -10 & 1 & -3 & 0 \end{array}\right]$$

Multiply the second row by $\frac{1}{5}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 4 & -10 & 1 & -3 & 0 \end{array}\right]$$

Add −4 times the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 0 & -2 & 1 & -\frac{7}{5} & -\frac{4}{5} \end{array}\right]$$

Multiply the third row by $-\frac{1}{2}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right]$$

Add −3 times the third row to the first and 2 times the third row to the second.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right]$$

$$\begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

15. $\left[\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right]$

Multiply the first row by −1.

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right]$$

Add −2 times the first row to the second and 4 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right]$$

Multiply the second row by $\frac{1}{10}$.

$$\left[\begin{array}{rrr|rrr} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 1 & -\frac{7}{10} & \frac{1}{5} & \frac{1}{10} & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right]$$

Add 10 times the second row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 1 & -\frac{7}{10} & \frac{1}{5} & \frac{1}{10} & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right]$$

Since there is a row of zeros on the left side,

$$\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix} \text{ is not invertible.}$$

17. $$\left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Add −1 times the first row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

Add −1 times the second row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right]$$

Multiply the third row by $-\dfrac{1}{2}$.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

Add −1 times the third row to both the first and second rows.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

19. $$\left[\begin{array}{rrr|rrr} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

Multiply the first row by $\dfrac{1}{2}$.

$$\left[\begin{array}{rrr|rrr} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

Add −2 times the first row to both the second and third rows.

$$\left[\begin{array}{rrr|rrr} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

Add −1 times the second row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Add −3 times the third row to the first.

$$\left[\begin{array}{rrr|rrr} 1 & 3 & 0 & \frac{1}{2} & 3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Add −3 times the second row to the first.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{7}{2} & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

21. $$\left[\begin{array}{rrrr|rrrr} 2 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right]$$

Interchange the first and second rows.

$$\left[\begin{array}{rrrr|rrrr} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 2 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right]$$

Add −2 times the first row to the second.

$$\left[\begin{array}{rrrr|rrrr} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right]$$

Interchange the second and fourth rows.

$$\left[\begin{array}{rrrr|rrrr} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \end{array}\right]$$

Multiply the second row by −1.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \end{array}\right]$$

Add 8 times the second row to the fourth.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 8 & 40 & 1 & -2 & 0 & -8 \end{array}\right]$$

Multiply the third row by $\frac{1}{2}$.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 8 & 40 & 1 & -2 & 0 & -8 \end{array}\right]$$

Add −8 times the third row to the fourth.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 40 & 1 & -2 & -4 & -8 \end{array}\right]$$

Multiply the fourth row by $\frac{1}{40}$.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right]$$

Add −5 times the fourth row to the second.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right]$$

Add −12 times the third row to the first and −4 times the third row to the second.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & -6 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right]$$

Add −2 times the second row to the first.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & -3 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right]$$

$$\begin{bmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & -3 & 0 \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{bmatrix}$$

23. $$\left[\begin{array}{cccc|cccc} -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 3 & -2 & 6 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$$

Multiply the first row by −1.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 2 & 3 & -2 & 6 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$$

Add −2 times the first row to the second.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 6 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$$

Interchange the second and third rows.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 6 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$$

Multiply the second row by −1.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & 6 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$$

Add −3 times the second row to the third.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 6 & 6 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$$

Interchange the third and fourth rows.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 6 & 6 & 2 & 1 & 3 & 0 \end{array}\right]$$

Add −6 times the third row to the fourth.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -24 & 2 & 1 & 3 & -6 \end{array}\right]$$

Multiply the fourth row by $-\frac{1}{24}$.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -\frac{1}{12} & -\frac{1}{24} & -\frac{1}{8} & \frac{1}{4} \end{array}\right]$$

Add −5 times the fourth row to the third.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & \frac{5}{12} & \frac{5}{24} & \frac{5}{8} & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & -\frac{1}{12} & -\frac{1}{24} & -\frac{1}{8} & \frac{1}{4} \end{array}\right]$$

Add the third row to the first and 2 times the third row to the second.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{7}{12} & \frac{5}{24} & \frac{5}{8} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{5}{6} & \frac{5}{12} & \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{5}{12} & \frac{5}{24} & \frac{5}{8} & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & -\frac{1}{12} & -\frac{1}{24} & -\frac{1}{8} & \frac{1}{4} \end{array}\right]$$

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{7}{12} & \frac{5}{24} & \frac{5}{8} & -\frac{1}{4} \\ \frac{5}{6} & \frac{5}{12} & \frac{1}{4} & -\frac{1}{2} \\ \frac{5}{12} & \frac{5}{24} & \frac{5}{8} & -\frac{1}{4} \\ -\frac{1}{12} & -\frac{1}{24} & -\frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

25. (a) $\left[\begin{array}{cccc|cccc} k_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 & 0 & 1 \end{array}\right]$

Multiply the first row by $\frac{1}{k_1}$, the second row by $\frac{1}{k_2}$, the third row by $\frac{1}{k^3}$, and the fourth row by $\frac{1}{k_4}$.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_4} \end{array}\right]$$

$$\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & \frac{1}{k_4} \end{bmatrix}$$

(b) $\left[\begin{array}{cccc|cccc} k & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

Multiply the first and third rows by $\frac{1}{k}$.

$$\left[\begin{array}{cccc|cccc} 1 & \frac{1}{k} & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{k} & 0 & 0 & \frac{1}{k} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Add $-\frac{1}{k}$ times the fourth row to the third and $-\frac{1}{k}$ times the second row to the first.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & -\frac{1}{k} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k} & -\frac{1}{k} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{k} & -\frac{1}{k} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{k} & -\frac{1}{k} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

27. $\left[\begin{array}{ccc|ccc} c & c & c & 1 & 0 & 0 \\ 1 & c & c & 0 & 1 & 0 \\ 1 & 1 & c & 0 & 0 & 1 \end{array}\right]$

Multiply the first row by $\frac{1}{c}$ (so $c \neq 0$ is a requirement).

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{c} & 0 & 0 \\ 1 & c & c & 0 & 1 & 0 \\ 1 & 1 & c & 0 & 0 & 0 \end{array}\right]$$

Add −1 times and the first row to the second and third rows.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{c} & 0 & 0 \\ 0 & c-1 & c-1 & -\frac{1}{c} & 1 & 0 \\ 0 & 0 & c-1 & -\frac{1}{c} & 0 & 1 \end{array}\right]$$

Multiply the second and third rows by $\dfrac{1}{c-1}$ (so $c \neq 1$ is a requirement).

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{c} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{c(c-1)} & \frac{1}{c-1} & 0 \\ 0 & 0 & 1 & -\frac{1}{c(c-1)} & 0 & \frac{1}{c-1} \end{array}\right]$$

Add −1 times the third row to the first and second rows.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{c-1} & 0 & -\frac{1}{c-1} \\ 0 & 1 & 0 & 0 & \frac{1}{c-1} & -\frac{1}{c-1} \\ 0 & 0 & 1 & -\frac{1}{c(c-1)} & 0 & \frac{1}{c-1} \end{array}\right]$$

The matrix is invertible for $c \neq 0, 1$.

29. Use elementary matrices to reduce $\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$ to I_2.

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}^{-1}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Note that other answers are possible.

31. Use elementary matrices to reduce $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ to I_3.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that other answers are possible.

33. From Exercise 29,

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix}$$

35. From Exercise 31,

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}$$

37. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}$

Add −1 times the first row to the second row.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix}$$

Add −1 times the first row to the third row.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 1 & -1 & 6 \end{bmatrix}$$

Add −1 times the second row to the first row.

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & -1 & 6 \end{bmatrix}$$

Add the second row to the third row.

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix} = B$$

True/False 1.5

(a) False; the product of two elementary matrices is not necessarily elementary.

(b) True; by Theorem 1.5.2.

(c) True; since A and B are row equivalent, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $B = E_k \cdots E_2E_1A$. Similarly there exist elementary matrices $E_1', E_2', \ldots, E_l'$ such that $C = E_l' \cdots E_2'E_1'B = E_l' \cdots E_2'E_1'E_k \cdots E_2E_1A$ so A and C are row equivalent.

(d) True; a homogeneous system has either exactly one solution or infinitely many solutions. Since A is not invertible, $A\mathbf{x} = 0$ cannot have exactly one solution.

(e) True; interchanging two rows is an elementary row operation, hence it does not affect whether a matrix is invertible.

(f) True; adding a multiple of the first row to the second row is an elementary row operation, hence it does not affect whether a matrix is invertible.

(g) False; since the sequence of row operations that convert an invertible matrix A into I_n is not unique, the expression of A as a product of elementary matrices is not unique. For instance,

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

(This is the matrix from Exercise 29.)

Section 1.6

Exercise Set 1.6

1. The matrix form of the system is

$$\begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}^{-1} = \frac{1}{6-5}\begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 12-9 \\ -10+9 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

The solution is $x_1 = 3, \; x_2 = -1$.

3. The matrix form of the system is

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & -3 & 4 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right]$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4+3 \\ 1+3 \\ 8-3-12 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -7 \end{bmatrix}$$

The solution is $x_1 = -1,\ x_2 = 4,\ x_3 = -7.$

5. The matrix form of the equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -4 \\ -4 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -4 & 0 & 1 & 0 \\ -4 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -5 & -1 & 1 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \\ 0 & 0 & -5 & -1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 0 & -5 & -1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{4}{5} & \frac{1}{5} & 0 \\ 0 & 1 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & 0 \end{array}\right]$$

$$\begin{bmatrix} \frac{1}{5} & 0 & -\frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3+2 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

The solution is $x = 1$, $y = 5$, $z = -1$.

7. The matrix form of the system is

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{6-5}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2b_1 - 5b_2 \\ -b_1 + 3b_2 \end{bmatrix}$$

The solution is $x_1 = 2b_1 - 5b_2,\ \ x_2 = -b_1 + 3b_2.$

9. Write and reduce the augmented matrix for both systems.

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 3 & 2 & 4 & 5 \end{array}\right]$$

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 0 & 17 & 1 & 11 \end{array}\right]$$

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 0 & 1 & \frac{1}{17} & \frac{11}{17} \end{array}\right]$$

$$\left[\begin{array}{cc|c|c} 1 & 0 & \frac{22}{17} & \frac{21}{17} \\ 0 & 1 & \frac{1}{17} & \frac{11}{17} \end{array}\right]$$

(i) The solution is $x_1 = \frac{22}{17}$, $x_2 = \frac{1}{17}$.

(ii) The solution is $x_1 = \frac{21}{17}$, $x_2 = \frac{11}{17}$.

11. Write and reduce the augmented matrix for the four systems.

$$\left[\begin{array}{cc|c|c|c|c} 4 & -7 & 0 & -4 & -1 & -5 \\ 1 & 2 & 1 & 6 & 3 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 2 & 1 & 6 & 3 & 1 \\ 4 & -7 & 0 & -4 & -1 & -5 \end{array}\right]$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 2 & 1 & 6 & 3 & 1 \\ 0 & -15 & -4 & -28 & -13 & -9 \end{array}\right]$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 2 & 1 & 6 & 3 & 1 \\ 0 & 1 & \frac{4}{15} & \frac{28}{15} & \frac{13}{15} & \frac{3}{5} \end{array}\right]$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 0 & \frac{7}{15} & \frac{34}{15} & \frac{19}{15} & -\frac{1}{5} \\ 0 & 1 & \frac{4}{15} & \frac{28}{15} & \frac{13}{15} & \frac{3}{5} \end{array}\right]$$

(i) The solution is $x_1 = \frac{7}{15}$, $x_2 = \frac{4}{15}$.

(ii) The solution is $x_1 = \frac{34}{15}$, $x_2 = \frac{28}{15}$.

(iii) The solution is $x_1 = \frac{19}{15}$, $x_2 = \frac{13}{15}$.

(iv) The solution is $x_1 = -\frac{1}{5}$, $x_2 = \frac{3}{5}$.

13. $\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ -2 & 1 & b_2 \end{array}\right]$

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 7 & 2b_1 + b_2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 1 & \frac{2}{7}b_1 + \frac{1}{7}b_2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{7}b_1 - \frac{3}{7}b_2 \\ 0 & 1 & \frac{2}{7}b_1 + \frac{1}{7}b_2 \end{array}\right]$$

There are no restrictions on b_1 and b_2.

15. $\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & -3 & 12 & 3b_1 + b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 1 & -4 & -\frac{4}{3}b_1 + \frac{1}{3}b_2 \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -\frac{5}{3}b_1 + \frac{2}{3}b_2 \\ 0 & 1 & -4 & -\frac{4}{3}b_1 + \frac{1}{3}b_2 \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right]$$

The only restriction is from the third row: $-b_1 + b_2 + b_3 = 0$ or $b_3 = b_1 - b_2$.

17. $$\left[\begin{array}{rrrr|l} 1 & -1 & 3 & 2 & b_1 \\ -2 & 1 & 5 & 1 & b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right]$$

$$\left[\begin{array}{rrrr|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ 0 & -1 & 11 & 5 & 3b_1+b_3 \\ 0 & 1 & -11 & -5 & -4b_1+b_4 \end{array}\right]$$

$$\left[\begin{array}{rrrr|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_4 \end{array}\right]$$

$$\left[\begin{array}{rrrr|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_4 \end{array}\right]$$

$$\left[\begin{array}{rrrr|c} 1 & 0 & -8 & -3 & -b_1-b_2 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_4 \end{array}\right]$$

From the bottom two rows, $b_1 - b_2 + b_3 = 0$ and $-2b_1 + b_2 + b_4 = 0.$

Thus $b_3 = -b_1 + b_2$ and $b_4 = 2b_1 - b_2$.

Expressing b_1 and b_2 in terms of b_3 and b_4 gives $b_1 = b_3 + b_4$ and $b_2 = 2b_3 + b_4$.

19. $$X = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{5} & \frac{4}{5} & -\frac{2}{5} & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 5 & -2 & 5 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & -1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right]$$

$$\begin{aligned} X &= \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 12 & -3 & 27 & 26 \\ -6 & -8 & 1 & -18 & -17 \\ -15 & -21 & 9 & -38 & -35 \end{bmatrix} \end{aligned}$$

21. Since $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, A is invertible. Thus, A^k is also invertible and $A^k\mathbf{x} = \mathbf{0}$ has only the trivial solution.
Note that $(A^k)^{-1} = (A^{-1})^k$.

23. Let x_1 be a fixed solution of $A\mathbf{x} = \mathbf{b}$, and let $\mathbf{x}$ be any other solution. Then
$A(\mathbf{x} - \mathbf{x}_1) = A\mathbf{x} - A\mathbf{x}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0}$. Thus $\mathbf{x}_0 = \mathbf{x} - \mathbf{x}_1$ is a solution to $A\mathbf{x} = \mathbf{0}$, so $\mathbf{x}$ can be expressed as $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_0$.
Also $A(\mathbf{x}_1 + \mathbf{x}_0) = A\mathbf{x}_1 + A\mathbf{x}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}$, so every matrix of the form $\mathbf{x}_1 + \mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$.

True/False 1.6

(a) True; if a system of linear equations has more than one solution, it has infinitely many solutions.

(b) True; if $A\mathbf{x} = \mathbf{b}$ has a unique solution, then A is invertible and $A\mathbf{x} = \mathbf{c}$ has the unique solution $A^{-1}\mathbf{c}$.

(c) True; if $AB = I_n$, then $B = A^{-1}$ and $BA = I_n$ also.

(d) True; elementary row operations do not change the solution set of a linear system, and row equivalent matrices can be obtained from one another by elementary row operations.

(e) True; since $(S^{-1}AS)\mathbf{x} = \mathbf{b}$, then $S(S^{-1}AS)\mathbf{x} = S\mathbf{b}$ or $A(S\mathbf{x}) = S\mathbf{b}$, $S\mathbf{x}$ is a solution of $A\mathbf{y} = S\mathbf{b}$.

(f) True; the system $A\mathbf{x} = 4\mathbf{x}$ is equivalent to the system $(A - 4I)\mathbf{x} = \mathbf{0}$. If the system $A\mathbf{x} = 4\mathbf{x}$ has a unique solution, then so will $(A - 4I)\mathbf{x} = \mathbf{0}$, hence $A - 4I$ is invertible. If $A - 4I$ is invertible, then the system $(A - 4I)\mathbf{x} = \mathbf{0}$ has a unique solution and so does the equivalent system $A\mathbf{x} = 4\mathbf{x}$.

(g) True; if AB were invertible, then both A and B would have to be invertible.

Section 1.7

Exercise Set 1.7

1. The matrix is a diagonal matrix with nonzero entries on the diagonal, so it is invertible.

The inverse is $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}$.

3. The matrix is a diagonal matrix with nonzero entries on the diagonal, so it is invertible.

The inverse is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

5. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 4 & -1 \\ 4 & 10 \end{bmatrix}$

7. $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} -3 & 2 & 0 & 4 & -4 \\ 1 & -5 & 3 & 0 & 3 \\ -6 & 2 & 2 & 2 & 2 \end{bmatrix}$

$= \begin{bmatrix} -15 & 10 & 0 & 20 & -20 \\ 2 & -10 & 6 & 0 & 6 \\ 18 & -6 & -6 & -6 & -6 \end{bmatrix}$

9. $A^2 = \begin{bmatrix} 1^2 & 0 \\ 0 & (-2)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$A^{-2} = \begin{bmatrix} 1^{-2} & 0 \\ 0 & (-2)^{-2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$

$A^{-k} = \begin{bmatrix} 1^{-k} & 0 \\ 0 & (-2)^{-k} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{(-2)^k} \end{bmatrix}$

11. $A^2 = \begin{bmatrix} \left(\frac{1}{2}\right)^2 & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^2 & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{16} \end{bmatrix}$

$A^{-2} = \begin{bmatrix} \left(\frac{1}{2}\right)^{-2} & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^{-2} & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^{-2} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$

$A^{-k} = \begin{bmatrix} \left(\frac{1}{2}\right)^{-k} & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^{-k} & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^{-k} \end{bmatrix}$

$= \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}$

13. The matrix is not symmetric since $a_{12} = -8 \neq 0 = a_{21}$.

15. The matrix is symmetric.

17. The matrix is not symmetric, since $a_{23} = -6 \neq 6 = a_{32}$.

19. The matrix is not symmetric, since $a_{13} = 1 \neq 3 = a_{31}$.

21. The matrix is not invertible because it is upper triangular and has a zero on the main diagonal.

23. For A to be symmetric, $a_{12} = a_{21}$ or $-3 = a + 5$, so $a = -8$.

25. For A to be invertible, the entries on the main diagonal must be nonzero. Thus $x \neq 1, -2, 4$.

27. By inspection, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

33. If $A^T A = A$, then
$A^T = (A^T A)^T = A^T (A^T)^T = A^T A = A$, so A is symmetric. Since $A = A^T$, then $A^2 = A^T A = A$.

35. **(a)** A is symmetric since $a_{ij} = a_{ji}$ for all i, j.

(b) $i^2 - j^2 = j^2 - i^2 \Rightarrow 2i^2 = 2j^2$ or
$i^2 = j^2 \Rightarrow i = j$, since $i, j > 0$
Thus, A is not symmetric unless $n = 1$.

(c) A is symmetric since $a_{ij} = a_{ji}$ for all i, j.

(d) Consider a_{12} and a_{21}.
$a_{12} = 2(1)^2 + 2(2)^3 = 18$ while
$a_{21} = 2(2)^2 + 2(1)^3 = 10$.
A is not symmetric unless $n = 1$.

37. **(a)** If A is invertible and skew-symmetric then $(A^{-1})^T = (A^T)^{-1} = (-A)^{-1} = -A^{-1}$ so A^{-1} is also skew-symmetric.

(b) Let A and B be skew-symmetric matrices.
$(A^T)^T = (-A)^T = -A^T$
$(A+B)^T = A^T + B^T = -A - B = -(A+B)$
$(A-B)^T = A^T - B^T = -A + B = -(A-B)$
$(kA)^T = k(A^T) = k(-A) = -kA$

(c) From the hint, it's sufficient to prove that $\frac{1}{2}(A + A^T)$ is symmetric and $\frac{1}{2}(A - A^T)$ is skew-symmetric.
$$\frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + (A^T)^T) = \frac{1}{2}(A + A^T)$$
Thus $\frac{1}{2}(A + A^T)$ is symmetric.
$$\begin{aligned} \frac{1}{2}(A - A^T)^T &= \frac{1}{2}(A^T - (A^T)^T) \\ &= \frac{1}{2}(A^T - A) \\ &= -\frac{1}{2}(A - A^T) \end{aligned}$$
Thus $\frac{1}{2}(A - A^T)$ is skew-symmetric.

39. From $A^T = -A$, the entries on the main diagonal must be zero and the reflections of entries across the main diagonal must be opposites.
$$A = \begin{bmatrix} 0 & 0 & -8 \\ 0 & 0 & -4 \\ 8 & 4 & 0 \end{bmatrix}$$

41. No; if A and B are commuting skew-symmetric matrices, then
$(AB)^T = (BA)^T = A^T B^T = (-A)(-B) = AB$ so the product of commuting skew-symmetric matrices is symmetric rather than skew-symmetric.

43. Let $A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$. Then
$A^3 = \begin{bmatrix} x^3 & y(x^2 + xz + z^2) \\ 0 & z^3 \end{bmatrix}$, so $x^3 = 1$ and $z^3 = -8$ or $x = 1$, $z = -2$. Then $3y = 30$ and $y = 10$.
$$A = \begin{bmatrix} 1 & 10 \\ 0 & -2 \end{bmatrix}$$

True/False 1.7

(a) True; since a diagonal matrix must be square and have zeros off the main diagonal, its transpose is also diagonal.

(b) False; the transpose of an upper triangular matrix is lower triangular.

(c) False; for example, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$.

(d) True; the entries above the main diagonal determine the entries below the main diagonal in a symmetric matrix.

(e) True; in an upper triangular matrix, the entries below the main diagonal are all zero.

(f) False; the inverse of an invertible lower triangular matrix is lower triangular.

(g) False; the diagonal entries may be negative, as long as they are nonzero.

(h) True; adding a diagonal matrix to a lower triangular matrix will not create nonzero entries above the main diagonal.

(i) True; since the entries below the main diagonal must be zero, so also must be the entries above the main diagonal.

(j) False; for example, $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}+\begin{bmatrix}5 & 0\\2 & 6\end{bmatrix}=\begin{bmatrix}6 & 2\\2 & 7\end{bmatrix}$ which is symmetric.

(k) False; for example, $\begin{bmatrix}1 & 2\\5 & 1\end{bmatrix}+\begin{bmatrix}1 & 3\\-5 & 1\end{bmatrix}=\begin{bmatrix}2 & 5\\0 & 2\end{bmatrix}$ which is upper triangular.

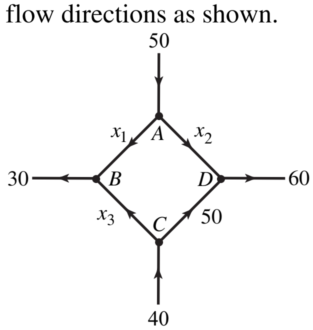
(l) False; for example $\begin{bmatrix}1 & 0\\1 & -1\end{bmatrix}^2=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}.$

(m) True; if $(kA)^T = kA,$ then $0=(kA)^T - kA = kA^T - kA = k(A^T - A)$ since $k \neq 0,$ then $A = A^T.$

Section 1.8

Exercise Set 1.8

1. Label the network with nodes, flow rates, and flow directions as shown.



Node A: $x_1 + x_2 = 50$

Node B: $x_1 + x_3 = 30$

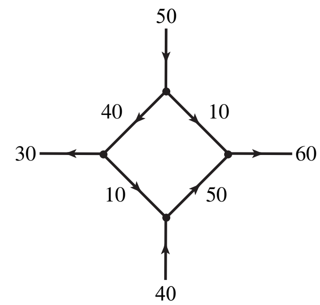
Node C: $x_3 + 50 = 40$

Node D: $x_2 + 50 = 60$

The linear system is

$$\begin{aligned} x_1 + x_2 &= 50\\ x_1 + x_3 &= 30\\ x_3 &= -10\\ x_2 &= 10 \end{aligned}$$

By inspection, the solution is $x_1 = 40,$ $x_2 = 10,$ $x_3 = -10$ and the completed network is shown.



3. (a) Consider the nodes from left to right on the top, then the bottom.

$300 + x_2 = 400 + x_3$
$750 + x_3 = 250 + x_4$
$100 + x_1 = 400 + x_2$
$200 + x_4 = 300 + x_1$

The system is

$$\begin{aligned} x_2 - x_3 &= 100\\ x_3 - x_4 &= -500\\ x_1 - x_2 &= 300\\ -x_1 + x_4 &= 100 \end{aligned}$$

(b) Reduce the augmented matrix.

$$\begin{bmatrix}0 & 1 & -1 & 0 & 100\\0 & 0 & 1 & -1 & -500\\1 & -1 & 0 & 0 & 300\\-1 & 0 & 0 & 1 & 100\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & 0 & 0 & 300\\0 & 0 & 1 & -1 & -500\\0 & 1 & -1 & 0 & 100\\-1 & 0 & 0 & 1 & 100\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & 0 & 0 & 300\\0 & 0 & 1 & -1 & -500\\0 & 1 & -1 & 0 & 100\\0 & -1 & 0 & 1 & 400\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & 0 & 0 & 300\\0 & 1 & -1 & 0 & 100\\0 & 0 & 1 & -1 & -500\\0 & -1 & 0 & 1 & 400\end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 300 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & -1 & 1 & 500 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 300 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & -1 & -400 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since x_4 is a free variable, let $x_4 = t$. The solution to the system is $x_1 = -100 + t$, $x_2 = -400 + t$, $x_3 = -500 + t$, $x_4 = t$.

(c) The flow from *A* to *B* is x_4. To keep the traffic flowing on all roads, each flow rate must be nonnegative. Thus, from $x_3 = -500 + t$, the minimum flow from *A* to *B* is 500 vehicles per hour. Thus $x_1 = 400$, $x_2 = 100$, $x_3 = 0$, $x_4 = 500$.

5. From Kirchoff's current law, applied at either of the nodes, we have $I_1 + I_2 = I_3$. From Kirchoff's voltage law, applied to the left and right loops, we have $2I_1 = 2I_2 + 6$ and $2I_2 + 4I_3 = 8$. Thus the currents I_1, I_2, and I_3 satisfy the following system of equations:

$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ 2I_1 - 2I_2 &= 6 \\ 2I_2 + 4I_3 &= 8 \end{aligned}$$

Reduce the augmented matrix.

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & -2 & 0 & 6 \\ 0 & 2 & 4 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & -2 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 11 \end{bmatrix}$$

From the last row we conclude that $I_3 = \frac{11}{5} = 2.2$ A, and from back substitution it then follows that $I_2 = 4 - 2I_3 = 4 - \frac{22}{5} = -\frac{2}{5} = -0.4$ A, and $I_1 = I_3 - I_2 = \frac{11}{5} + \frac{2}{5} = \frac{13}{5} = 2.6$ A. Since I_2 is negative, its direction is opposite to that indicated in the figure.

7. From application of the current law at each of the four nodes (clockwise from upper left) we have $I_1 = I_2 + I_4$, $I_4 = I_3 + I_5$, $I_6 = I_3 + I_5$, and $I_1 = I_2 + I_6$. These four equations can be reduced to the following:

$I_1 = I_2 + I_4$, $I_4 = I_3 + I_5$, $I_4 = I_6$

From application of the voltage law to the three inner loops (left to right) we have $10 = 20I_1 + 20I_2$, $20I_2 = 20I_3$, and $10 + 20I_3 = 20I_5$. These equations can be simplified to:

$2I_1 + 2I_2 = 1$, $I_3 = I_2$, $2I_5 - 2I_3 = 1$

This gives us six equations in the unknown currents $I_1, \ldots, I_6$. But, since $I_3 = I_2$ and $I_6 = I_4$, this can be simplified to a system of only four equations in the variables I_1, I_2, I_4, and I_5:

$$\begin{aligned} I_1 - I_2 - I_4 &= 0 \\ I_2 - I_4 + I_5 &= 0 \\ 2I_1 + 2I_2 &= 1 \\ -2I_2 + 2I_5 &= 1 \end{aligned}$$

Reduce the augmented matrix.

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 \\ 0 & -2 & 0 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 4 & 2 & 0 & 1 \\ 0 & -2 & 0 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 6 & -4 & 1 \\ 0 & 0 & -2 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 4 & 1 \\ 0 & 0 & 6 & -4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -\frac{1}{2} \\ 0 & 0 & 6 & -4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -\frac{1}{2} \\ 0 & 0 & 0 & 8 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

From this we conclude that $I_5 = \frac{1}{2}$,

$I_6 = I_4 = -\frac{1}{2} + 2I_5 = -\frac{1}{2} + 1 = \frac{1}{2}$,

$I_3 = I_2 = I_4 - I_5 = \frac{1}{2} - \frac{1}{2} = 0$, and

$I_1 = I_2 + I_4 = 0 + \frac{1}{2} = \frac{1}{2}$.

9. We seek positive integers x_1, x_2, x_3, and x_4 that will balance the chemical equation
$x_1(C_3H_8) + x_2(O_2) \to x_3(CO_2) + x_4(H_2O)$
For each of the atoms in the equation (C, H, and O), the number of atoms on the left must be equal to the number of atoms on the right. This leads to the equations $3x_1 = x_3$, $8x_1 = 2x_4$, and $2x_2 = 2x_3 + x_4$. These equations form the homogeneous linear system.

$$\begin{aligned} 3x_1 \qquad - x_3 \qquad &= 0 \\ 8x_1 \qquad\qquad - 2x_4 &= 0 \\ 2x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

Reduce the augmented matrix.

$$\begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & \frac{8}{3} & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{8}{3} & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} & 0 \end{bmatrix}$$

From this we conclude that x_4 is a free variable, and that the general solution of the system is given by $x_4 = t$, $x_3 = \frac{3}{4}x_4 = \frac{3}{4}t$,

$x_2 = x_3 + \frac{1}{2}x_4 = \frac{3}{4}t + \frac{1}{2}t = \frac{5}{4}t$, and

$x_1 = \frac{1}{3}x_3 = \frac{1}{4}t$. The smallest positive integer solutions are obtained by taking $t = 4$, in which case we obtain $x_1 = 1$, $x_2 = 5$, $x_3 = 3$, and $x_4 = 4$. Thus the balanced equation is $C_3H_8 + 5O_2 \to 3CO_2 + 4H_2O$.

11. We must find positive integers x_1, x_2, x_3, and x_4 that balance the chemical equation

$$x_1(CH_3COF) + x_2(H_2O) \to x_3(CH_3COOH) + x_4(HF)$$

For each of the elements in the equation (C, H, O, and F), the number of atoms on the left must equal the number of atoms on the right. This leads to the equations $x_1 = x_3$, $3x_1 + 2x_2 = 4x_3 + x_4$, $x_1 + x_2 = 2x_3$, and $x_1 = x_4$. This is a very simple system of equations having (by inspection) the general solution $x_1 = x_2 = x_3 = x_4 = t$.

Thus, taking $t = 1$, the balanced equation is $CH_3COF + H_2O \to CH_3COOH + HF$.

13. The graph of the polynomial $p(x) = a_0 + a_1x + a_2x^2$ passes through the points (1, 1), (2, 2), and (3, 5) if and only if the coefficients a_0, a_1, and a_2 satisfy the equations

$$\begin{aligned} a_0 + a_1 + a_2 &= 1 \\ a_0 + 2a_1 + 4a_2 &= 2 \\ a_0 + 3a_1 + 9a_2 &= 5 \end{aligned}$$

Reduce the augmented matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 1 & 3 & 9 & 5 \end{bmatrix}$$

Add −1 times row 1 to rows 2 and 3.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 8 & 4 \end{bmatrix}$$

Add −2 times row 2 to row 3.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

We conclude that $a_2 = 1$, and from back substitution it follows that $a_1 = 1 - 3a_2 = -2$, and $a_0 = 1 - a_1 - a_2 = 2$. Thus the quadratic polynomial is $p(x) = x^2 - 2x + 2$.

15. The graph of the polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ passes through the points (−1, −1), (0, 1), (1, 3), and (4, −1), if and only if the coefficients a_0, a_1, a_2, and a_3 satisfy the equations

$$\begin{aligned} a_0 - a_1 + a_2 - a_3 &= -1 \\ a_0 &= 1 \\ a_0 + a_1 + a_2 + a_3 &= 3 \\ a_0 + 4a_1 + 16a_2 + 64a_3 &= -1 \end{aligned}$$

The augmented matrix of this system is

$$\begin{bmatrix} 1 & -1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 3 \\ 1 & 4 & 16 & 64 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 3 \\ 1 & 4 & 16 & 64 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 4 & 16 & 64 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 4 & 16 & 64 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 20 & 60 & -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 20 & 60 & -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 60 & -10 \end{bmatrix}$$

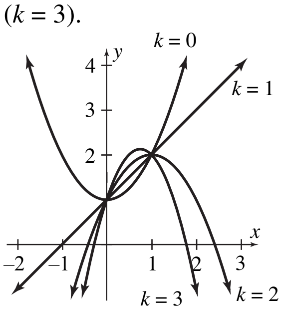
From the first row we see that $a_0 = 1$, and from the last two rows we conclude that $a_2 = 0$ and $a_3 = -\frac{1}{6}$. Finally, from back substitution, it follows that $a_1 = 2 + a_2 - a_3 = \frac{13}{6}$. Thus the polynomial is $p(x) = -\frac{1}{6}x^3 + \frac{13}{6}x + 1$.

17. (a) The quadratic polynomial $p(x) = a_0 + a_1x + a_2x^2$ passes through the points (0, 1) and (1, 2) if and only if the coefficients a_0, a_1, and a_2 satisfy the equations

$$a_0 = 1$$
$$a_0 + a_1 + a_2 = 2$$

The general solution of this system, using a_1 as a parameter, is $a_0 = 1$, $a_1 = k$, $a_2 = 1 - k$. Thus this family of polynomials is represented by the equation $p(x) = 1 + kx + (1-k)x^2$ where $-\infty < k < \infty$.

(b) Below are graphs of four curves in the family: $y = 1 + x^2$ $(k = 0)$, $y = 1 + x$ $(k = 1)$, $y = 1 + 2x - x^2$ $(k = 2)$, and $y = 1 + 3x - 2x^2$ $(k = 3)$.



True/False 1.8

(a) True; only such networks were considered.

(b) False; when a current passes through a resistor, there is a drop in electrical potential in the circuit.

(c) True

(d) False; the number of atoms of *each element* must be the same on each side of the equation.

(e) False; this is only true if the n points have distinct x-coordinates.

Section 1.9

Exercise Set 1.9

1. (a) A consumption matrix for this economy is

$$C = \begin{bmatrix} 0.50 & 0.25 \\ 0.25 & 0.10 \end{bmatrix}.$$

(b) We have $(I - C)^{-1} = \begin{bmatrix} 0.50 & -0.25 \\ -0.25 & 0.90 \end{bmatrix}^{-1}$

$$= \frac{1}{0.3875}\begin{bmatrix} 0.90 & 0.25 \\ 0.25 & 0.50 \end{bmatrix},$$

and so the production needed to provide customers \$7000 worth of mechanical work and \$14,000 worth of body work is given by

$$\mathbf{x} = (I - C)^{-1}\mathbf{d}$$
$$= \frac{1}{0.3875}\begin{bmatrix} 0.9 & 0.25 \\ 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 7000 \\ 14{,}000 \end{bmatrix}$$
$$\approx \begin{bmatrix} \$25{,}290 \\ \$22{,}581 \end{bmatrix}$$

3. (a) A consumption matrix for this economy is

$$C = \begin{bmatrix} 0.1 & 0.6 & 0.4 \\ 0.3 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0.2 \end{bmatrix}.$$

(b) The Leontief equation $(I - C)\mathbf{x} = \mathbf{d}$ is represented by the augmented matrix

$$\left[\begin{array}{rrr|r} 0.9 & -0.6 & -0.4 & 1930 \\ -0.3 & 0.8 & -0.3 & 3860 \\ -0.4 & -0.1 & 0.8 & 5790 \end{array}\right].$$

The reduced row echelon form of this matrix is

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 31{,}500 \\ 0 & 1 & 0 & 26{,}500 \\ 0 & 0 & 1 & 26{,}300 \end{array}\right].$$

Thus the required production vector is

$$\mathbf{x} = \begin{bmatrix} \$31{,}500 \\ \$26{,}500 \\ \$26{,}300 \end{bmatrix}.$$

5. The Leontief equation $(I - C)\mathbf{x} = \mathbf{d}$ for this economy is $\begin{bmatrix} 0.9 & -0.3 \\ -0.5 & 0.6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 50 \\ 60 \end{bmatrix}$; thus the required production vector is

$$\mathbf{x} = (I - C)^{-1}\mathbf{d}$$
$$= \frac{1}{0.39}\begin{bmatrix} 0.6 & 0.3 \\ 0.5 & 0.9 \end{bmatrix}\begin{bmatrix} 50 \\ 60 \end{bmatrix}$$
$$= \frac{1}{0.39}\begin{bmatrix} 48 \\ 79 \end{bmatrix}$$
$$\approx \begin{bmatrix} 123.08 \\ 202.56 \end{bmatrix}.$$

7. (a) The Leontief equation for this economy is $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$. If $d_1 = 2$ and $d_2 = 0$, the system is consistent with general solution $x_1 = 4$, $x_2 = t$ $(0 \le t < \infty)$. If $d_1 = 2$ and $d_2 = 1$ the system is inconsistent.

(b) The consumption matrix $C = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ indicates that the entire output of the second sector is consumed in producing that output; thus there is nothing left to satisfy any outside demand. Mathematically, the Leontief matrix $I - C = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

9. Since $c_{21}c_{12} < 1 - c_{11}$, it follows that $(1 - C_{11}) - C_{21}C_{12} > 0$. From this we conclude that the matrix $I - C$ is invertible, and $(I - C)^{-1} = \frac{1}{(1 - C_{11}) - C_{21}C_{12}}\begin{bmatrix} 1 & c_{12} \\ c_{21} & 1 - c_{11} \end{bmatrix}$ has nonnegative entries. This shows that the economy is productive. Thus the Leontief equation $(I - C)\mathbf{x} = \mathbf{d}$ has a unique solution for every demand vector $\mathbf{d}$.

11. If C has row sums less than 1, then C^T has column sums less than 1. Thus, from Theorem 1.9.1, it follows that $I - C^T$ is invertible and that $(I - C^T)^{-1}$ has nonnegative entries. Since $I - C = (I^T - C^T)^T = (I - C^T)^T$, we can conclude that $I - C$ is invertible and that $(I - C)^{-1} = ((I - C^T)^T)^{-1} = ((I - C^T)^{-1})^T$ has nonnegative entries.

True/False 1.9

(a) False; open sectors are those that do not produce output.

(b) True

(c) True

(d) True; by Theorem 1.9.1.

(e) True

Chapter 1 Supplementary Exercises

1. The corresponding system is

$$\begin{aligned} 3x_1 - x_2 + 4x_4 &= 1 \\ 2x_1 + 3x_3 + 3x_4 &= -1 \end{aligned}$$

$$\begin{bmatrix} 3 & -1 & 0 & 4 & 1 \\ 2 & 0 & 3 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 & \frac{4}{3} & \frac{1}{3} \\ 2 & 0 & 3 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 & \frac{4}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & 3 & \frac{1}{3} & -\frac{5}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & \frac{9}{2} & \frac{1}{2} & -\frac{5}{2} \end{bmatrix}$$

Thus, x_3 and x_4 are free variables.

Let $x_3 = s$ and $x_4 = t$.

$$x_2 = -\frac{9}{2}x_3 - \frac{1}{2}x_4 - \frac{5}{2} = -\frac{9}{2}s - \frac{1}{2}t - \frac{5}{2}$$

$$\begin{aligned} x_1 &= \frac{1}{3}x_2 - \frac{4}{3}x_4 + \frac{1}{3} \\ &= \frac{1}{3}\left(-\frac{9}{2}s - \frac{1}{2}t - \frac{5}{2}\right) - \frac{4}{3}t + \frac{1}{3} \\ &= -\frac{3}{2}s - \frac{3}{2}t - \frac{1}{2} \end{aligned}$$

The solution is $x_1 = -\frac{3}{2}s - \frac{3}{2}t - \frac{1}{2}$, $x_2 = -\frac{9}{2}s - \frac{1}{2}t - \frac{5}{2}$, $x_3 = s$, $x_4 = t$.

3. The corresponding system is

$$\begin{aligned} 2x_1 - 4x_2 + x_3 &= 6 \\ -4x_1 + 3x_3 &= -1 \\ x_2 - x_3 &= 3 \end{aligned}$$

$$\begin{bmatrix} 2 & -4 & 1 & 6 \\ -4 & 0 & 3 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ -4 & 0 & 3 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ 0 & -8 & 5 & 11 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & 3 \\ 0 & -8 & 5 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -3 & 35 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -\frac{35}{3} \end{bmatrix}$$

Thus $x_3 = -\dfrac{35}{3}$, $x_2 = x_3 + 3 = -\dfrac{26}{3}$, and

$x_1 = 2x_2 - \dfrac{1}{2}x_3 + 3 = -\dfrac{52}{3} + \dfrac{35}{6} + 3 = -\dfrac{17}{2}$.

The solution is $x_1 = -\dfrac{17}{2}$, $x_2 = -\dfrac{26}{3}$,

$x_3 = -\dfrac{35}{3}$.

5. Reduce the augmented matrix.

$$\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & x \\ \frac{4}{5} & \frac{3}{5} & y \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3}x \\ \frac{4}{5} & \frac{3}{5} & y \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3}x \\ 0 & \frac{5}{3} & -\frac{4}{3}x + y \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3}x \\ 0 & 1 & -\frac{4}{5}x + \frac{3}{5}y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{3}{5}x + \frac{4}{5}y \\ 0 & 1 & -\frac{4}{5}x + \frac{3}{5}y \end{bmatrix}$$

The solution is $x' = \dfrac{3}{5}x + \dfrac{4}{5}y$, $y' = -\dfrac{4}{5}x + \dfrac{3}{5}y$.

7. Reduce the augmented matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 1 & 5 & 10 & 44 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 4 & 9 & 35 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & \frac{9}{4} & \frac{35}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{5}{4} & \frac{1}{4} \\ 0 & 1 & \frac{9}{4} & \frac{35}{4} \end{bmatrix}$$

Thus z is a free variable. Let $z = s$. Then $y = -\dfrac{9}{4}s + \dfrac{35}{4}$ and $x = \dfrac{5}{4}s + \dfrac{1}{4}$. For x, y, and z to be positive integers, s must be a positive integer such that $5s + 1$ and $-9s + 35$ are positive and divisible by 4. Since $-9s + 35 > 0$, $s < 4$, so $s = 1, 2$, or 3. The only possibility is $s = 3$. Thus, $x = 4$, $y = 2$, $z = 3$.

9. $$\begin{bmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{bmatrix}$$

Add −1 times the first row to the second.

$$\begin{bmatrix} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & a & 2 & b \end{bmatrix}$$

Add −1 times the second row to the third.

$$\begin{bmatrix} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{bmatrix}$$

(a) The system has a unique solution if the corresponding matrix can be put in reduced row echelon form without division by zero. Thus, $a \neq 0$, $b \neq 2$ is required.

(b) If $a \neq 0$ and $b = 2$, then x_3 is a free variable and the system has a one-parameter solution.

(c) If $a = 0$ and $b = 2$, the system has a two-parameter solution.

(d) If $b \neq 2$ but $a = 0$, the system has no solution.

9. Note that K must be a 2×2 matrix. Let
$K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
Then
$$\begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 6 & -6 \\ 6 & -1 & 1 \\ -4 & 0 & 0 \end{bmatrix}$$
or $\begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2a & b & -b \\ 2c & d & -d \end{bmatrix} = \begin{bmatrix} 8 & 6 & -6 \\ 6 & -1 & 1 \\ -4 & 0 & 0 \end{bmatrix}$ or
$$\begin{bmatrix} 2a+8c & b+4d & -b-4d \\ -4a+6c & -2b+3d & 2b-3d \\ 2a-4c & b-2d & -b+2d \end{bmatrix} = \begin{bmatrix} 8 & 6 & -6 \\ 6 & -1 & 1 \\ -4 & 0 & 0 \end{bmatrix}.$$
Thus
$$\begin{aligned} 2a \quad + 8c \quad &= 8 \\ b \quad + 4d &= 6 \\ -4a \quad + 6c \quad &= 6 \\ -2b \quad + 3d &= -1 \\ 2a \quad - 4c \quad &= -4 \\ b \quad - 2d &= 0 \end{aligned}$$
Note that we have omitted the 3 equations obtained by equating elements of the last columns of these matrices because the information so obtained would be just a repeat of that gained by equating elements of the second columns. The augmented matrix of the above system is
$$\begin{bmatrix} 2 & 0 & 8 & 0 & 8 \\ 0 & 1 & 0 & 4 & 6 \\ -4 & 0 & 6 & 0 & 6 \\ 0 & -2 & 0 & 3 & -1 \\ 2 & 0 & -4 & 0 & -4 \\ 0 & 1 & 0 & -2 & 0 \end{bmatrix}.$$
The reduced row-echelon form of this matrix is
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
Thus $a = 0$, $b = 2$, $c = 1$, and $d = 1$.

The matrix is $K = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$.

13. (a) Here X must be a 2×3 matrix. Let
$X = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$.
Then
$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 5 \end{bmatrix} \text{ or}$$
$$\begin{bmatrix} -a+b+3c & b+c & a-c \\ -d+e+3f & e+f & d-f \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 5 \end{bmatrix}.$$
Equating entries in the first row leads to the system
$$\begin{aligned} -a+b+3c &= 1 \\ b \ + c &= 2 \\ a \quad - c &= 0 \end{aligned}$$
Equating entries in the second row yields the system
$$\begin{aligned} -d+e+3f &= -3 \\ e \ + f &= 1 \\ d \quad - f &= 5 \end{aligned}$$
These systems can be solved simultaneously by reducing the following augmented matrix.
$$\left[\begin{array}{rrr|r|r} -1 & 1 & 3 & 1 & -3 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 0 & 5 \end{array}\right]$$
$$\left[\begin{array}{rrr|r|r} 1 & 0 & -1 & 0 & 5 \\ 0 & 1 & 1 & 2 & 1 \\ -1 & 1 & 3 & 1 & -3 \end{array}\right]$$
$$\left[\begin{array}{rrr|r|r} 1 & 0 & -1 & 0 & 5 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 \end{array}\right]$$
$$\left[\begin{array}{rrr|r|r} 1 & 0 & -1 & 0 & 5 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$$
$$\left[\begin{array}{rrr|r|r} 1 & 0 & 0 & -1 & 6 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$$
Thus, $a = -1$, $b = 3$, $c = -1$, $d = 6$, $e = 0$, $f = 1$ and $X = \begin{bmatrix} -1 & 3 & -1 \\ 6 & 0 & 1 \end{bmatrix}$.

(b) X must be a 2×2 matrix.

Let $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Then

$$X\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x+3y & -x & 2x+y \\ z+3w & -z & 2z+w \end{bmatrix}.$$

If we equate matrix entries, this gives us the equations

$$\begin{aligned} x+3y &= -5 & z+3w &= 6 \\ -x &= -1 & -z &= -3 \\ 2x+y &= 0 & 2z+w &= 7 \end{aligned}$$

Thus $x = 1$ and $z = 3$, so that the top two equations give $y = -2$ and $w = 1$. Since these values are consistent with the bottom two equations, we have that

$$X = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}.$$

(c) Again, X must be a 2×2 matrix. Let $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, so that the matrix equation becomes

$$\begin{bmatrix} 3x+z & 3y+w \\ -x+2z & -y+2w \end{bmatrix} - \begin{bmatrix} x+2y & 4x \\ z+2w & 4z \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 5 & 4 \end{bmatrix}.$$

This yields the system of equations

$$\begin{aligned} 2x-2y+z \qquad &= 2 \\ -4x+3y \qquad +w &= -2 \\ -x \qquad +z-2w &= 5 \\ -y-4z+2w &= 4 \end{aligned}$$

with matrix $\begin{bmatrix} 2 & -2 & 1 & 0 & 2 \\ -4 & 3 & 0 & 1 & -2 \\ -1 & 0 & 1 & -2 & 5 \\ 0 & -1 & -4 & 2 & 4 \end{bmatrix}$ which

reduces to $\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{113}{37} \\ 0 & 1 & 0 & 0 & -\frac{160}{37} \\ 0 & 0 & 1 & 0 & -\frac{20}{37} \\ 0 & 0 & 0 & 1 & -\frac{46}{37} \end{bmatrix}$.

Hence, $x = -\dfrac{113}{37}$, $y = -\dfrac{160}{37}$, $z = -\dfrac{20}{37}$,

$w = -\dfrac{46}{37}$, and $X = \begin{bmatrix} -\frac{113}{37} & -\frac{160}{37} \\ -\frac{20}{37} & -\frac{46}{37} \end{bmatrix}$.

15. Since the coordinates of the given points must satisfy the polynomial, we have

$$\begin{aligned} p(1) = 2 \quad &\Rightarrow \quad a+b+c = 2 \\ p(-1) = 6 \quad &\Rightarrow \quad a-b+c = 6 \\ p(2) = 3 \quad &\Rightarrow \quad 4a+2b+c = 3 \end{aligned}$$

This system has augmented matrix

$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 6 \\ 4 & 2 & 1 & 3 \end{bmatrix}$ which reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Thus, $a = 1$, $b = -2$, and $c = 3$.

19. First suppose that $AB^{-1} = B^{-1}A$. Note that all matrices must be square and of the same size. Therefore $(AB^{-1})B = (B^{-1}A)B$ or $A = B^{-1}AB$ so that $BA = B(B^{-1}AB) = (BB^{-1})(AB) = AB$. A similar argument shows that if $AB = BA$ then $AB^{-1} = B^{-1}A$.

21. Consider the ith row of AB. By the definition of matrix multiplication, this is

$$\begin{aligned} &a_{i1}b_{11} + a_{i2}b_{21} + \cdots + a_{in}b_{n1} \\ &= \frac{a_{i1} + a_{i2} + \cdots + a_{in}}{n} \\ &= \overline{r_i} \end{aligned}$$

since all entries $b_{i1} = \dfrac{1}{n}$.

Chapter 10

Applications of Linear Algebra

Section 10.1

Exercise Set 10.1

1. (a) Substituting the coordinates of the points into Equation (4) yields $\begin{vmatrix} x & y & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$ which, upon cofactor expansion along the first row, yields $-3x + y + 4 = 0$; that is, $y = 3x - 4$.

(b) As in (a), $\begin{vmatrix} x & y & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0$ yields $2x + y - 1 = 0$ or $y = -2x + 1$.

2. (a) Equation (9) yields $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 40 & 2 & 6 & 1 \\ 4 & 2 & 0 & 1 \\ 34 & 5 & 3 & 1 \end{vmatrix} = 0$ which, upon first-row cofactor expansion, yields $18(x^2 + y^2) - 72x - 108y + 72 = 0$ or, dividing by 18, $x^2 + y^2 - 4x - 6y + 4 = 0$. Completing the squares in x and y yields the standard form $(x-2)^2 + (y-3)^2 = 9$.

(b) As in (a), $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 8 & 2 & -2 & 1 \\ 34 & 3 & 5 & 1 \\ 52 & -4 & 6 & 1 \end{vmatrix} = 0$ yields $50(x^2 + y^2) + 100x - 200y - 1000 = 0$; that is, $x^2 + y^2 + 2x - 4y - 20 = 0$. In standard form this is $(x+1)^2 + (y-2)^2 = 25$.

3. Using Equation (10) we obtain $\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 4 & 0 & 0 & 2 & 0 & 1 \\ 4 & -10 & 25 & 2 & -5 & 1 \\ 16 & -4 & 1 & 4 & -1 & 1 \end{vmatrix} = 0$ which is the same as $\begin{vmatrix} x^2 & xy & y^2 & x & y \\ 0 & 0 & 1 & 0 & -1 \\ 4 & 0 & 0 & 2 & 0 \\ 4 & -10 & 25 & 2 & -5 \\ 16 & -4 & 1 & 4 & -1 \end{vmatrix} = 0$ by expansion along the second row (taking advantage of the zeros there). Add column five to column three and take advantage of another row of all but one zero to get $\begin{vmatrix} x^2 & xy & y^2+y & x \\ 4 & 0 & 0 & 2 \\ 4 & -10 & 20 & 2 \\ 16 & -4 & 0 & 4 \end{vmatrix} = 0.$

Now expand along the first row and get $160x^2 + 320xy + 160(y^2 + y) - 320x = 0$; that is, $x^2 + 2xy + y^2 - 2x + y = 0$, which is the equation of a parabola.

4. (a) From Equation (11), the equation of the plane is $\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & -3 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & 1 \end{vmatrix} = 0.$

Expansion along the first row yields $-2x - 4y - 2z = 0$; that is, $x + 2y + z = 0$.

(b) As in (a), $\begin{vmatrix} x & y & z & 1 \\ 2 & 3 & 1 & 1 \\ 2 & -1 & -1 & 1 \\ 1 & 2 & 1 & 1 \end{vmatrix} = 0$ yields $-2x + 2y - 4z + 2 = 0$; that is $-x + y - 2z + 1 = 0$ for the equation of the plane.

5. (a) Equation (11) involves the determinant of the coefficient matrix of the system

$$\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Rows 2 through 4 show that the plane passes through the three points (x_i, y_i, z_i), $i = 1, 2, 3$, while row 1 gives the equation $c_1x + c_2y + c_3z + c_4 = 0$. For the plane passing through the origin parallel to the plane passing through the three points, the constant term in the final equation will be 0, which is accomplished by using

$$\begin{vmatrix} x & y & z & 0 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

(b) The parallel planes passing through the origin are $x + 2y + z = 0$ and $-x + y - 2z = 0$, respectively.

6. (a) Using Equation (12), the equation of the sphere is $\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 14 & 1 & 2 & 3 & 1 \\ 6 & -1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 6 & 1 & 2 & -1 & 1 \end{vmatrix} = 0.$

Expanding by cofactors along the first row yields

$16(x^2 + y^2 + z^2) - 32x - 64y - 32z + 32 = 0;$

that is, $(x^2 + y^2 + z^2) - 2x - 4y - 2x + 2 = 0.$

Completing the squares in each variable yields the standard form

$(x-1)^2 + (y-2)^2 + (z-1)^2 = 4.$

Note: When evaluating the cofactors, it is useful to take advantage of the column of ones and elementary row operations; for example, the cofactor of $x^2 + y^2 + z^2$ above can be evaluated as follows:

$$\begin{vmatrix} 1 & 2 & 3 & 1 \\ -1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 1 \\ -2 & 0 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -4 & 0 \end{vmatrix} = 16 \text{ by}$$

cofactor expansion of the latter determinant along the last column.

(b) As in (a), $\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 5 & 0 & 1 & -2 & 1 \\ 11 & 1 & 3 & 1 & 1 \\ 5 & 2 & -1 & 0 & 1 \\ 11 & 3 & 1 & -1 & 1 \end{vmatrix} = 0$

yields

$-24(x^2 + y^2 + z^2) + 48x + 48y + 72 = 0;$ that is, $x^2 + y^2 + z^2 - 2x - 2y - 3 = 0$ or in standard form, $(x-1)^2 + (y-1)^2 + z^2 = 5.$

7. Substituting each of the points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), and (x_5, y_5) into the equation

$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6 = 0$ yields

$$\begin{array}{c} c_1x_1^2 + c_2x_1y_1 + c_3y_1^2 + c_4x_1 + c_5y_1 + c_6 = 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ c_1x_5^2 + c_2x_5y_5 + c_3y_5^2 + c_4x_5 + c_5y_5 + c_6 = 0. \end{array}$$

These together with the original equation form a homogeneous linear system with a non-trivial solution for $c_1, c_2, \ldots, c_6$. Thus the determinant of the coefficient matrix is zero, which is exactly Equation (10).

8. As in the previous problem, substitute the coordinates (x_i, y_i, z_i) of each of the three points into the equation $c_1x + c_2y + c_3z + c_4 = 0$ to obtain a homogeneous system with nontrivial solution for $c_1, \ldots, c_4$. Thus the determinant of the coefficient matrix is zero, which is exactly Equation (11).

9. Substituting the coordinates (x_i, y_i, z_i) of the four points into the equation $c_1(x^2 + y^2 + z^2) + c_2x + c_3y + c_4z + c_5 = 0$ of the sphere yields four equations, which together with the above sphere equation form a homogeneous linear system for $c_1, \ldots, c_5$ with a nontrivial solution. Thus the determinant of this system is zero, which is Equation (12).

10. Upon substitution of the coordinates of the three points (x_1, y_1), (x_2, y_2) and (x_3, y_3), we obtain the equations:

$$c_1 y + c_2 x^2 + c_3 x + c_4 = 0$$
$$c_1 y_1 + c_2 x_1^2 + c_3 x_1 + c_4 = 0$$
$$c_1 y_2 + c_2 x_2^2 + c_3 x_2 + c_4 = 0$$
$$c_1 y_3 + c_2 x_3^2 + c_3 x_3 + c_4 = 0.$$

This is a homogeneous system with a nontrivial solution for c_1, c_2, c_3, c_4, so the determinant of the coefficient matrix is zero; that is,

$$\begin{vmatrix} y & x^2 & x & 1 \\ y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ y_3 & x_3^2 & x_3 & 1 \end{vmatrix} = 0.$$

11. Expanding the determinant in Equation (9) by cofactors of the first row makes it apparent that the coefficient of $x^2 + y^2$ in the final equation is $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. If the points are collinear, then the columns are linearly dependent $(y_i = mx_i + b)$, so the coefficient of $x^2 + y^2 = 0$ and the resulting equation is that of the line through the three points.

12. If the three distinct points are collinear then two of the coordinates can be expressed in terms of the third. Without loss of generality, we can say that y and z can be expressed in terms of x, i.e., x is the parameter. If the line is $(x, ax + b, cx + d)$, then the determinant in Equation (11) is equivalent to $\begin{vmatrix} x & -ax-b & -cx-d & 1 \\ x_1 & 0 & 0 & 1 \\ x_2 & 0 & 0 & 1 \\ x_3 & 0 & 0 & 1 \end{vmatrix}$.

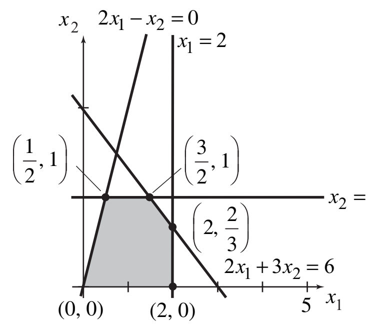
Expanding along the first row, it is clear that the determinant is 0 and Equation (11) becomes $0 = 0$.

13. As in Exercise 11, the coefficient of $x^2 + y^2 + z^2$ will be 0, so Equation (12) gives the equation of the plane in which the four points lie.

Section 10.2

Exercise Set 10.2

1.

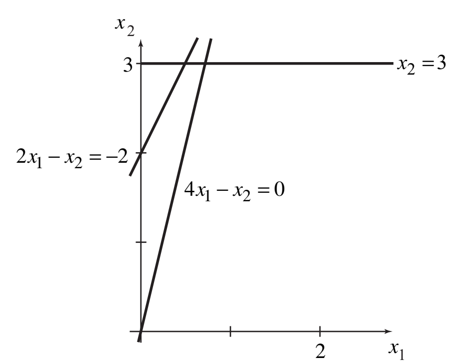


In the figure the feasible region is shown and the extreme points are labeled. The values of the objective function are shown in the following table:

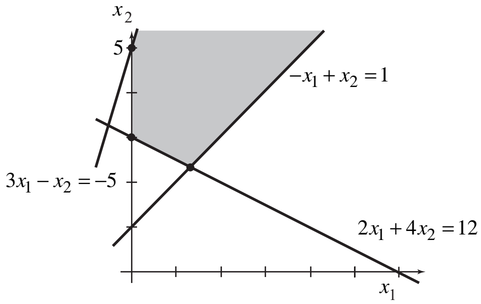
Extreme point (x_1, x_2)	Value of $z = 3x_1 + 2x_2$
(0, 0)	0
$\left(\frac{1}{2}, 1\right)$	$\frac{7}{2}$
$\left(\frac{3}{2}, 1\right)$	$\frac{13}{2}$
$\left(2, \frac{2}{3}\right)$	$\frac{22}{3}$
(2, 0)	6

Thus the maximum, $\frac{22}{3}$, is attained when $x_1 = 2$ and $x_2 = \frac{2}{3}$.

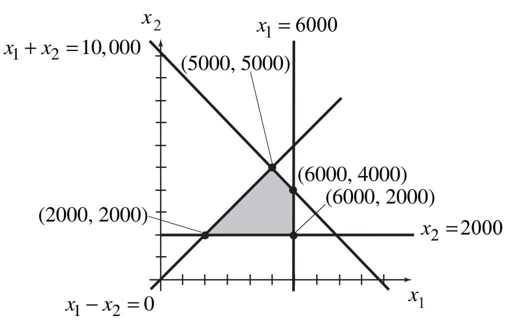
2.



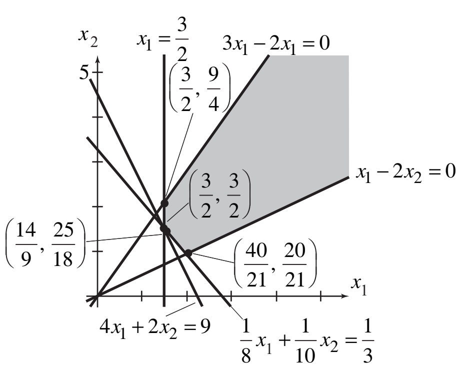
The intersection of the five half-planes defined by the constraints is empty. Thus this problem has no feasible solutions.

3. 

The feasible region for this problem, shown in the figure, is unbounded. The value of $z = -3x_1 + 2x_2$ cannot be minimized in this region since it becomes arbitrarily negative as we travel outward along the line $-x_1 + x_2 = 1$; i.e., the value of z is $-3x_1 + 2x_2 = -3x_1 + 2(x_1 + 1) = -x_1 + 2$ and x_1 can be arbitrarily large.

4. 

The feasible region and vertices for this problem are shown in the figure. The maximum value of the objective function $z = .1x_1 + .07x_2$ is attained when $x_1 = 6000$ and $x_2 = 4000$, so that $z = 880$. In other words, she should invest \$6000 in bond A and \$4000 in bond B for annual yield of \$880.

5. 

The feasible region and its extreme points are shown in the figure. Though the region is unbounded, x_1 and x_2 are always positive, so the objective function $z = 7.5x_1 + 5.0x_2$ is also. Thus, it has a minimum, which is attained at the point where $x_1 = \frac{14}{9}$ and $x_2 = \frac{25}{18}$. The value of z there is $\frac{335}{18}$. In the problem's terms, if we use $\frac{7}{9}$ cup of milk and $\frac{25}{18}$ ounces of corn flakes, a minimum cost of 18.6¢ is realized.

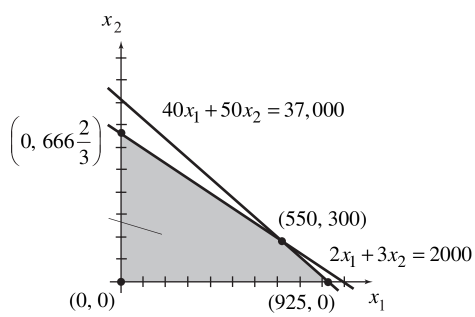
6. **(a)** Both $x_1 \geq 0$ and $x_2 \geq 0$ are nonbinding. Of the remaining constraints, only $2x_1 + 3x_2 \leq 24$ is binding.

(b) $x_1 - x_2 \leq v$ will be binding if the line $x_1 - x_2 = v$ intersects the line $x_2 = 6$ with $0 < x_1 < 3$, thus if $-6 < v < -3$. If $v < -6$, then the feasible region is empty.

(c) $x_2 \leq v$ will be nonbinding for $v \geq 8$ and the feasible region will be empty for $v < 0$.

7. Letting x_1 be the number of Company *A*'s containers shipped and x_2 the number of Company *B*'s, the problem is:
Maximize $z = 2.20x_1 + 3.00x_2$ subject to

$$\begin{aligned} 40x_1 + 50x_2 &\leq 37,000 \\ 2x_1 + 3x_2 &\leq 2000 \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$



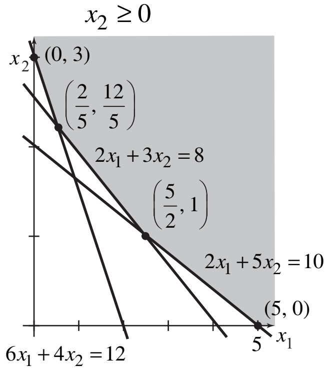
The feasible region is shown in the figure. The vertex at which the maximum is attained is $x_1 = 550$ and $x_2 = 300$, where $z = 2110$. A truck should carry 550 containers from Company *A* and 300 containers from Company *B* for maximum shipping charges of \$2110.

8. We must now maximize $z = 2.50x_1 + 3.00x_2$. The feasible region is as in Exercise 7, but now the maximum occurs for $x_1 = 925$ and $x_2 = 0$, where $z = 2312.50$.

925 containers from Company *A* and no containers from Company *B* should be shipped for maximum shipping charges of $2312.50.

9. Let x_1 be the number of pounds of ingredient *A* used and x_2 the number of pounds of ingredient *B*. Then the problem is:
Minimize $z = 8x_1 + 9x_2$ subject to

$$\begin{aligned} 2x_1 + 5x_2 &\geq 10 \\ 2x_1 + 3x_2 &\geq 8 \\ 6x_1 + 4x_2 &\geq 12 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$



Though the feasible region shown in the figure is unbounded, the objective function is always positive there and hence must have a minimum. This minimum occurs at the vertex where $x_1 = \frac{2}{5}$ and $x_2 = \frac{12}{5}$. The minimum value of z is $\frac{124}{5}$ or 24.8¢.

Each sack of feed should contain 0.4 lb of ingredient *A* and 2.4 lb of ingredient *B* for a minimum cost of 24.8¢.

10. It is sufficient to show that if a linear function has the same value at two points, then it has that value along the line connecting the two points. Let $ax_1 + bx_2$ be the function, and $ax_1' + bx_2' = ax_1'' + bx_2'' = c$. Then $(tx_1' + (1-t)x_1'', tx_2' + (1-t)x_2'')$ is an arbitrary point on the line connecting (x_1', x_2') to (x_1'', x_2'') and

$$\begin{aligned} a(tx_1' + (1-t)x_1'') + b(tx_2' + (1-t)x_2'') &= t(ax_1' + bx_2') + (1-t)(ax_1'' + bx_2'') \\ &= tc + (1-t)c \\ &= c. \end{aligned}$$

Section 10.3

Exercise Set 10.3

1. The number of oxen is 50 per herd, and there are 7 herds, so there are 350 oxen. Hence the total number of oxen and sheep is 350 + 350 = 700.

2. (a) The equations are $B = 2A$, $C = 3(A + B)$, $D = 4(A + B + C)$, $300 = A + B + C + D$. Solving this linear system gives $A = 5$ (and $B = 10$, $C = 45$, $D = 240$).

(b) The equations are $B = 2A$, $C = 3B$, $D = 4C$, $132 = A + B + C + D$. Solving this linear system gives $A = 4$ (and $B = 8$, $C = 24$, $D = 96$).

3. Note that this is, effectively, Gaussian elimination applied to the augmented matrix $\begin{bmatrix} 1 & 1 & 10 \\ 1 & \frac{1}{4} & 7 \end{bmatrix}$.

4. (a) Let x represent oxen and y represent sheep, then the equations are $5x + 2y = 10$ and $2x + 5y = 8$. The corresponding array is

2	5
5	2
8	10

and the elimination step subtracts twice column 2 from five times column 1, giving

	5
21	2
20	10

and so $y = \frac{20}{21}$ unit for a sheep, and $x = \frac{34}{21}$ units for an ox.

(b) Let x, y, and z represent the number of bundles of each class. Then the equations are

$$\begin{aligned} 2x + y \quad &= 1 \\ 3y + z &= 1 \\ x \quad 4z &= 1 \end{aligned}$$

and the corresponding array is

	2	1
3	1	
1		4
1	1	1

Subtract two times the numbers in the third column from the second column to get

		1
3	1	
1	−8	4
1	−1	1

Now subtract three times the numbers in the second column from the first column to get

		1
	1	
25	−8	4
4	−1	1

This is equivalent to the linear system

$$\begin{aligned} x+4z &= 1 \\ y-8z &= -1. \\ 25z &= 4 \end{aligned}$$

From this, the solution is that a bundle of the first class contains $\frac{9}{25}$ measure, second class contains $\frac{7}{25}$ measure, and third class contains $\frac{4}{25}$ measure.

5. (a) From equations 2 through n, $x_j = a_j - x_1$ ($j = 2, ..., n$). Using these equations in equation 1 gives

$$x_1 + (a_2 - x_1) + (a_3 - x_1) + \cdots + (a_n - x_1) = a_1$$

$$x_1 = \frac{a_2 + a_3 + \cdots + a_n - a_1}{n-2}.$$

First find x_1 in terms of the known quantities n and the a_i. Then we can use $x_j = a_j - x_1$ ($j = 2, ..., n$) to find the other x_i.

(b) Exercise 7(b) may be solved using this technique. x_1 represents gold, x_2 represents brass, x_3 represents tin, and x_4 represents much-wrought iron, so $n = 4$ and

$a_1 = 60,$

$a_2 = \frac{2}{3}(60) = 40,$

$a_3 = \frac{3}{4}(60) = 45,$

$a_4 = \frac{3}{5}(60) = 36.$

$$\begin{aligned} x_1 &= \frac{(a_2 + a_3 + a_4) - a_1}{n-2} \\ &= \frac{40 + 45 + 36 - 60}{4-2} \\ &= \frac{61}{2} \end{aligned}$$

$$x_2 = a_2 - x_1 = 40 - \frac{61}{2} = \frac{19}{2}$$

$$x_3 = a_3 - x_1 = 45 - \frac{61}{2} = \frac{29}{2}$$

$$x_4 = a_4 - x_1 = 36 - \frac{61}{2} = \frac{11}{2}$$

The crown was made with $30\frac{1}{2}$ minae of gold, $9\frac{1}{2}$ minae of brass, $14\frac{1}{2}$ minae of tin, and $5\frac{1}{2}$ minae of iron.

6. (a) We can write this as

$$5x + y + z - K = 0$$
$$x + 7y + z - K = 0$$

$x + y + 8 - K = 0$ (a 3×4 system). Since the coefficient matrix of equations (5) is invertible (its determinant is 262), there is a unique solution x, y, z for every K; hence, K is an arbitrary parameter.

(b) Gaussian elimination gives $x = \frac{21K}{131}$, $y = \frac{14K}{131}$, $z = \frac{12K}{131}$, for any choice of K. Since 131 is prime, we must choose K to be an integer multiple of 131 to get integer solutions. The obvious choice is $K = 131$, giving $x = 21$, $y = 14$, $z = 12$, and $K = 131$.

(c) This solution corresponds to $K = 262 = 2 \cdot 131$.

7. (a) The system is $x + y = 1000$, $\left(\frac{1}{5}\right)x - \left(\frac{1}{4}\right)y = 10$, with solution $x = 577$ and $\frac{7}{9}$, $y = 422$ and $\frac{2}{9}$. The legitimate son receives $577\frac{7}{9}$ staters, the illegitimate son receives $422\frac{2}{9}$.

(b) The system is $G + B = \left(\frac{2}{3}\right)60$, $G + T = \left(\frac{3}{4}\right)60$, $G + I = \left(\frac{3}{5}\right)60$, $G + B + T + I = 60$, with solution $G = 30.5$, $B = 9.5$, $T = 14.5$, and $I = 5.5$. The crown was made with $30\frac{1}{2}$ minae of gold, $9\frac{1}{2}$ minae of brass, $14\frac{1}{2}$ minae of tin, and $5\frac{1}{2}$ minae of iron.

(c) The system is $A = B + \left(\frac{1}{3}C,\right)$ $B = C + \left(\frac{1}{3}\right)A$, $C = \left(\frac{1}{3}\right)B + 10$, with solution $A = 45$, $B = 37.5$, and $C = 22.5$. The first person has 45 minae, the second has $37\frac{1}{2}$, and the third has $22\frac{1}{2}$.

Section 10.4

Exercise Set 10.4

2. (a) Set $h = .2$ and

$x_1 = 0,\ y_1 = .00000$
$x_2 = .2,\ y_2 = .19867$
$x_3 = .4,\ y_3 = .38942$
$x_4 = .6,\ y_4 = .56464$
$x_5 = .8,\ y_5 = .71736$
$x_6 = 1.0,\ y_6 = .84147$

Then

$$\frac{6(y_1 - 2y_2 + y_3)}{h^2} = -1.1880$$
$$\frac{6(y_2 - 2y_3 + y_4)}{h^2} = -2.3295$$
$$\frac{6(y_3 - 2y_4 + y_5)}{h^2} = -3.3750$$
$$\frac{6(y_4 - 2y_5 + y_6)}{h^2} = -4.2915$$

and the linear system (21) for the parabolic runout spline becomes

$$\begin{bmatrix} 5 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ M_5 \end{bmatrix} = \begin{bmatrix} -1.1880 \\ -2.3295 \\ -3.3750 \\ -4.2915 \end{bmatrix}.$$

Solving this system yields $M_2 = -.15676$, $M_3 = -.40421$, $M_4 = -.55592$, $M_5 = -.74712$.

From (19) and (20) we have $M_1 = M_2 = -.15676$, $M_6 = M_5 = -.74712$.

The specific interval $.4 \le x \le .6$ is the third interval. Using (14) to solve for a_3, b_3, c_3, and d_3 gives

$$a_3 = \frac{(M_4 - M_3)}{6h} = -.12643$$
$$b_3 = \frac{M_3}{2} = -.20211$$
$$c_3 = \frac{(y_4 - y_3)}{h} - \frac{(M_4 + 2M_3)h}{6} = .92158$$
$$d_3 = y_3 = .38942.$$

The interpolating parabolic runout spline for $.4 \le x \le .6$ is thus

$$S(x) = -.12643(x - .4)^3 - .20211(x - .4)^2 + .92158(x - .4) + .38942.$$

(b) $S(.5) = -.12643(.1)^3 - .20211(.1)^2 + .92158(.1) + .38942 = .47943.$

Since $\sin(.5) = S(.5) = .47943$ to five decimal places, the percentage error is zero.

3. (a) Given that the points lie on a single cubic curve, the cubic runout spline will agree exactly with the single cubic curve.

(b) Set $h = 1$ and
$x_1 = 0,\ y_1 = 1$
$x_2 = 1,\ y_2 = 7$
$x_3 = 2,\ y_3 = 27$
$x_4 = 3,\ y_4 = 79$
$x_5 = 4,\ y_5 = 181.$
Then
$$\frac{6(y_1 - 2y_2 + y_3)}{h^2} = 84$$
$$\frac{6(y_2 - 2y_3 + y_4)}{h^2} = 192$$
$$\frac{6(y_3 - 2y_4 + y_5)}{h^2} = 300$$
and the linear system (24) for the cubic runout spline becomes
$$\begin{bmatrix} 6 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} 84 \\ 192 \\ 300 \end{bmatrix}.$$
Solving this system yields $M_2 = 14$, $M_3 = 32,\ M_4 = 50.$
From (22) and (23) we have
$M_1 = 2M_2 - M_3 = -4$
$M_5 = 2M_4 - M_3 = 68.$
Using (14) to solve for the a_i's, b_i's, c_i's, and d_i's we have $a_1 = \frac{(M_2 - M_1)}{6h} = 3,$
$$a_2 = \frac{(M_3 - M_2)}{6h} = 3,\quad a_3 = \frac{(M_4 - M_3)}{6h} = 3,$$
$$a_4 = \frac{(M_5 - M_4)}{6h} = 3,$$

$$b_1 = \frac{M_1}{2} = -2,\quad b_2 = \frac{M_2}{2} = 7,$$
$$b_3 = \frac{M_3}{2} = 16,\quad b_4 = \frac{M_4}{2} = 25,$$

$$c_1 = \frac{(y_2 - y_1)}{h} - \frac{(M_2 + 2M_1)h}{6} = 5,$$
$$c_2 = \frac{(y_3 - y_2)}{h} - \frac{(M_3 + 2M_2)h}{6} = 10,$$
$$c_3 = \frac{(y_4 - y_3)}{h} - \frac{(M_4 + 2M_3)h}{6} = 33,$$
$$c_4 = \frac{(y_5 - y_4)}{h} - \frac{(M_5 + 2M_4)h}{6} = 74,$$

$d_1 = y_1 = 1,\ d_2 = y_2 = 7,\ d_3 = y_3 = 27,$
$d_4 = y_4 = 79.$
For $0 \le x \le 1$ we have
$S(x) = S_1(x) = 3x^3 - 2x^2 + 5x + 1.$
For $1 \le x \le 2$ we have
$$\begin{aligned} S(x) &= S_2(x) \\ &= 3(x-1)^3 + 7(x-1)^2 + 10(x-1) + 7 \\ &= 3x^3 - 2x^2 + 5x + 1. \end{aligned}$$
For $2 \le x \le 3$ we have
$$\begin{aligned} S(x) &= S_3(x) \\ &= 3(x-2)^3 + 16(x-2)^2 + 33(x-2) + 27 \\ &= 3x^3 - 2x^2 + 5x + 1. \end{aligned}$$
For $3 \le x \le 4$ we have
$$\begin{aligned} S(x) &= S_4(x) \\ &= 3(x-3)^3 + 25(x-3)^2 + 74(x-3) + 79 \\ &= 3x^3 - 2x^2 + 5x + 1. \end{aligned}$$
Thus $S_1(x) = S_2(x) = S_3(x) = S_4(x)$, or
$S(x) = 3x^3 - 2x^2 + 5x + 1$ for $0 \le x \le 4$.

4. The linear system (16) for the natural spline becomes $\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} -.0001116 \\ -.0000816 \\ -.0000636 \end{bmatrix}.$
Solving this system yields $M_2 = -.0000252$, $M_3 = -.0000108,\ M_4 = -.0000132.$
From (17) and (18) we have $M_1 = 0,\ M_5 = 0.$
Solving for the a_i's, b_i's, c_i's, and d_i's from Equations (14) we have
$$a_1 = \frac{(M_2 - M_1)}{6h} = -.00000042,$$
$$a_2 = \frac{(M_3 - M_2)}{6h} = .00000024,$$
$$a_3 = \frac{(M_4 - M_3)}{6h} = -.00000004,$$
$$a_4 = \frac{(M_5 - M_4)}{6h} = -.00000022,$$

$$b_1 = \frac{M_1}{2} = 0,\quad b_2 = \frac{M_2}{2} = -.0000126,$$
$$b_3 = \frac{M_3}{2} = -.0000054,\quad b_4 = \frac{M_4}{2} = -.0000066,$$
$$b_5 = \frac{M_5}{2} = 0.$$

$$c_1 = \frac{(y_2 - y_1)}{h} - \frac{(M_2 + 2M_1)h}{6} = .000214,$$

$c_2 = \dfrac{(y_3 - y_2)}{h} - \dfrac{(M_3 + 2M_2)h}{6} = .000088,\;\; c_3 = \dfrac{(y_4 - y_3)}{h} - \dfrac{(M_4 + 2M_3)h}{6} = -.000092,$

$c_4 = \dfrac{(y_5 - y_4)}{h} - \dfrac{(M_5 + 2M_4)h}{6} = -.000212,$

$d_1 = y_1 = .99815,\;\; d_2 = y_2 = .99987,\;\; d_3 = y_3 = .99973,\;\; d_4 = y_4 = .99823.$

The resulting natural spline is

$$S(x) = \begin{cases} -.00000042(x+10)^3 + .000214(x+10) + .99815, & -10 \le x \le 0 \\ .00000024(x)^3 - .0000126(x)^2 + .000088(x) + .99987, & 0 \le x \le 10 \\ -.00000004(x-10)^3 - .0000054(x-10)^2 - .000092(x-10) + .99973, & 10 \le x \le 20 \\ -.00000022(x-20)^3 - .0000066(x-20)^2 - .000212(x-20) + .99823, & 20 \le x \le 30. \end{cases}$$

Assuming the maximum is attained in the interval [0, 10] we set $S'(x)$ equal to zero in this interval:

$S'(x) = .00000072x^2 - .0000252x + .000088 = 0.$

To three significant digits the root of this quadratic equation in the interval [0, 10] is $x = 3.93$, and $S(3.93) = 1.00004$.

5. The linear system (24) for the cubic runout spline becomes $\begin{bmatrix} 6 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} -.0001116 \\ -.0000816 \\ -.0000636 \end{bmatrix}.$

Solving this system yields $M_2 = -.0000186,\;\; M_3 = -.0000131,\;\; M_4 = -.0000106.$

From (22) and (23) we have $M_1 = 2M_2 - M_3 = -.0000241,\;\; M_5 = 2M_4 - M_3 = -.0000081.$

Solving for the a_i's, b_i's, c_i's, and d_i's from Equations (14) we have $a_1 = \dfrac{(M_2 - M_1)}{6h} = .00000009,$

$a_2 = \dfrac{(M_3 - M_2)}{6h} = .00000009,\;\; a_3 = \dfrac{(M_4 - M_3)}{6h} = .00000004,\;\; a_4 = \dfrac{(M_5 - M_4)}{6h} = .00000004.$

$b_1 = \dfrac{M_1}{2} = -.0000121,\;\; b_2 = \dfrac{M_2}{2} = -.0000093,\;\; b_3 = \dfrac{M_3}{2} = -.0000066,\;\; b_4 = \dfrac{M_4}{2} = -.0000053,$

$c_1 = \dfrac{(y_2 - y_1)}{h} - \dfrac{(M_2 + 2M_1)h}{6} = .000282,\;\; c_2 = \dfrac{(y_3 - y_2)}{h} - \dfrac{(M_3 + 2M_2)h}{6} = .000070,$

$c_3 = \dfrac{(y_4 - y_3)}{h} - \dfrac{(M_4 + 2M_3)h}{6} = -.000087,\;\; c_4 = \dfrac{(y_5 - y_4)}{h} - \dfrac{(M_5 + 2M_4)h}{6} = -.000207,$

$d_1 = y_1 = .99815,\;\; d_2 = y_2 = .99987,\;\; d_3 = y_3 = .99973,\;\; d_4 = y_4 = .99823.$

The resulting cubic runout spline is

$$S(x) = \begin{cases} .00000009(x+10)^3 - .0000121(x+10)^2 + .000282(x+10) + .99815, & -10 \le x \le 0 \\ .00000009(x)^3 - .0000093(x)^2 + .000070(x) + .99987, & 0 \le x \le 10 \\ .00000004(x-10)^3 - .0000066(x-10)^2 - .000087(x-10) + .99973, & 10 \le x \le 20 \\ .00000004(x-20)^3 - .0000053(x-20)^2 - .000207(x-20) + .99823, & 20 \le x \le 30. \end{cases}$$

Assuming the maximum is attained in the interval [0, 10], we set $S'(x)$ equal to zero in this interval:

$S'(x) = .00000027x^2 - .0000186x + .000070 = 0.$

To three significant digits the root of this quadratic equation in the interval [0, 10] is 4.00 and $S(4.00) = 1.00001$.

6. (a) Set $h = .5$ and

$x_1 = 0, \quad y_1 = 0$

$x_2 = .5, \; y_2 = 1$

$x_3 = 1, \; y_3 = 0$

For a natural spline with $n = 3$,

$M_1 = M_3 = 0$ and

$\dfrac{6(y_1 - 2y_2 + y_3)}{h^2} = -48 = 4M_2$. Thus

$M_2 = -12.$

$a_1 = \dfrac{M_2 - M_1}{6h} = -4, \; a_2 = \dfrac{M_3 - M_2}{6h} = 4,$

$b_1 = \dfrac{M_1}{2} = 0, \; b_2 = \dfrac{M_2}{2} = -6,$

$c_1 = \dfrac{y_2 - y_1}{h} - \dfrac{(M_2 + 2M_1)h}{6} = 3,$

$c_2 = \dfrac{y_3 - y_2}{h} - \dfrac{(M_3 + 2M_2)h}{6} = 0, \; d_1 = 0,$

$d_2 = 1$

For $0 \le x \le .5$, we have

$S(x) = S_1(x) = -4x^3 + 3x.$

For $.5 \le x \le 1$, we have

$$S(x) = S_2(x) = 4(x - .5)^3 - 6(x - .5)^2 + 1 = 4x^3 - 12x^2 + 9x - 1.$$

The resulting natural spline is

$$S(x) = \begin{cases} -4x^3 + 3x & 0 \le x \le 0.5 \\ 4x^3 - 12x^2 + 9x - 1 & 0.5 \le x \le 1 \end{cases}.$$

(b) Again $h = .5$ and

$x_1 = .5, \; y_1 = 1$

$x_2 = 1, \; y_2 = 0$

$x_3 = 1.5, \; y_3 = -1$

$4M_2 = \dfrac{6(y_1 - 2y_2 + y_3)}{h^2} = 0$

Thus $M_1 = M_2 = M_3 = 0$, hence all a_i and b_i are also 0.

$c_1 = \dfrac{y_2 - y_1}{h} = -2, \; c_2 = \dfrac{y_3 - y_2}{h} = -2$

$d_1 = y_1 = 1, \; d_2 = y_2 = 0$

For $.5 \le x \le 1$, we have

$S(x) = S_1(x) = -2(x - .5) + 1 = -2x + 2.$

For $1 \le x \le 1.5$, we have

$S(x) = S_2(x) = -2(x - 1) + 0 = -2x + 2.$

The resulting natural spline is

$$S(x) = \begin{cases} -2x + 2 & 0.5 \le x \le 1 \\ -2x + 2 & 1 \le x \le 1.5 \end{cases}.$$

(c) The three data points are collinear, so the spline is just the line the points lie on.

7. (b) Equations (15) together with the three equations in part (a) of the exercise statement give

$$\begin{aligned}
4M_1 + M_2 + M_{n-1} &= \frac{6(y_{n-1} - 2y_1 + y_2)}{h^2} \\
M_1 + 4M_2 + M_3 &= \frac{6(y_1 - 2y_2 + y_3)}{h^2} \\
M_2 + 4M_3 + M_4 &= \frac{6(y_2 - 2y_3 + y_4)}{h^2} \\
&\vdots \\
M_{n-3} + 4M_{n-2} + M_{n-1} &= \frac{6(y_{n-3} - 2y_{n-2} + y_{n-1})}{h^2} \\
M_1 + M_{n-2} + 4M_{n-1} &= \frac{6(y_{n-2} - 2y_{n-1} + y_1)}{h^2}.
\end{aligned}$$

The linear system for $M_1, M_2, \ldots, M_{n-1}$ in matrix form is

$$\begin{bmatrix} 4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 4 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_{n-1} - 2y_1 + y_2 \\ y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ \vdots \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_1 \end{bmatrix}.$$

8. (b) Equations (15) together with the two equations in part (a) give

$$\begin{aligned}
2M_1 + M_2 &= \frac{6(y_2 - y_1 - hy_1')}{h^2}\\
M_1 + 4M_2 + M_3 &= \frac{6(y_1 - 2y_2 + y_3)}{h^2}\\
M_2 + 4M_3 + M_4 &= \frac{6(y_2 - 2y_3 + y_4)}{h^2}\\
&\vdots\\
M_{n-2} + 4M_{n-1} + M_n &= \frac{6(y_{n-2} - 2y_{n-1} + y_n)}{h^2}\\
M_{n-1} + 2M_n &= \frac{6(y_{n-1} - y_n + hy_n')}{h^2}.
\end{aligned}$$

This linear system for $M_1, M_2, \ldots, M_n$ in matrix form is

$$\begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 1 & 4 & 1 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 4 & \cdots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 4 & 1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} M_1\\ M_2\\ M_3\\ \vdots\\ M_{n-2}\\ M_{n-1}\\ M_n \end{bmatrix} = \frac{6}{h^2}\begin{bmatrix} -hy_1' - y_1 + y_2\\ y_1 - 2y_2 + y_3\\ y_2 - 2y_3 + y_4\\ \vdots\\ y_{n-2} - 2y_{n-1} + y_n\\ y_{n-1} - y_n + hy_n' \end{bmatrix}.$$

Section 10.5

Exercise Set 10.5

1. (a) $\mathbf{x}^{(1)} = P\mathbf{x}^{(0)} = \begin{bmatrix} .4\\ .6 \end{bmatrix}$, $\mathbf{x}^{(2)} = P\mathbf{x}^{(1)} = \begin{bmatrix} .46\\ .54 \end{bmatrix}$.

Continuing in this manner yields

$\mathbf{x}^{(3)} = \begin{bmatrix} .454\\ .546 \end{bmatrix}$.

$\mathbf{x}^{(4)} = \begin{bmatrix} .4546\\ .5454 \end{bmatrix}$ and $\mathbf{x}^{(5)} = \begin{bmatrix} .45454\\ .54546 \end{bmatrix}$.

(b) P is regular because all of the entries of P are positive. Its steady-state vector $\mathbf{q}$ solves $(I - P)\mathbf{q} = \mathbf{0}$; that is, $\begin{bmatrix} .6 & -.5\\ -.6 & .5 \end{bmatrix}\begin{bmatrix} q_1\\ q_2 \end{bmatrix} = \begin{bmatrix} 0\\ 0 \end{bmatrix}$.

This yields one independent equation, $.6q_1 - .5q_2 = 0$, or $q_1 = \frac{5}{6}q_2$. Solutions are thus of the form $\mathbf{q} = s\begin{bmatrix} \frac{5}{6}\\ 1 \end{bmatrix}$. Set

$s = \frac{1}{\frac{5}{6}+1} = \frac{6}{11}$ to obtain $\mathbf{q} = \begin{bmatrix} \frac{5}{11}\\ \frac{6}{11} \end{bmatrix}$.

2. (a) $\mathbf{x}^{(1)} = P\mathbf{x}^{(0)} = \begin{bmatrix} .7\\ .2\\ .1 \end{bmatrix}$; likewise $\mathbf{x}^{(2)} = \begin{bmatrix} .23\\ .52\\ .25 \end{bmatrix}$ and $\mathbf{x}^{(3)} = \begin{bmatrix} .273\\ .396\\ .331 \end{bmatrix}$.

(b) P is regular because all of its entries are positive. To solve $(I - P)\mathbf{q} = \mathbf{0}$, i.e.

$\begin{bmatrix} .8 & -.1 & -.7\\ -.6 & .6 & -.2\\ -.2 & -.5 & .9 \end{bmatrix}\begin{bmatrix} q_1\\ q_2\\ q_3 \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}$, reduce the coefficient matrix to row-echelon form:

$$\begin{bmatrix} 8 & -1 & -7\\ -6 & 6 & -2\\ -2 & -5 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 5 & -9\\ 0 & -21 & 29\\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{22}{21}\\ 0 & 1 & -\frac{29}{21}\\ 0 & 0 & 0 \end{bmatrix}.$$

This yields solutions (setting $q_3 = s$) of the form $\begin{bmatrix} \frac{22}{21}\\ \frac{29}{21}\\ 1 \end{bmatrix}s$.

To obtain a probability vector, take

$s = \frac{1}{\frac{22}{21} + \frac{29}{21} + 1} = \frac{21}{72}$, yielding $\mathbf{q} = \begin{bmatrix} \frac{22}{72}\\ \frac{29}{72}\\ \frac{21}{72} \end{bmatrix}$.

3. (a) Solve $(I - P)\mathbf{q} = \mathbf{0}$, i.e.,

$\begin{bmatrix} \frac{2}{3} & -\frac{3}{4}\\ -\frac{2}{3} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} q_1\\ q_2 \end{bmatrix} = \begin{bmatrix} 0\\ 0 \end{bmatrix}$.

The only independent equation is

$\frac{2}{3}q_1 = \frac{3}{4}q_2$, yielding $\mathbf{q} = \begin{bmatrix} \frac{9}{8} \\ 1 \end{bmatrix} s$. Setting $s = \frac{8}{17}$ yields $\mathbf{q} = \begin{bmatrix} \frac{9}{17} \\ \frac{8}{17} \end{bmatrix}$.

(b) As in (a), solve $\begin{bmatrix} .19 & -.26 \\ -.19 & .26 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ i.e., $.19q_1 = .26q_2$. Solutions have the form $\mathbf{q} = \begin{bmatrix} \frac{26}{19} \\ 1 \end{bmatrix} s$. Set $s = \frac{19}{45}$ to get $\mathbf{q} = \begin{bmatrix} \frac{26}{45} \\ \frac{19}{45} \end{bmatrix}$.

(c) Again, solve $\begin{bmatrix} \frac{2}{3} & -\frac{1}{2} & 0 \\ -\frac{1}{3} & 1 & -\frac{1}{4} \\ -\frac{1}{3} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ by reducing the coefficient matrix to row-echelon form: $\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$ yielding solutions of the form $\mathbf{q} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{3} \\ 1 \end{bmatrix} s$.

Set $s = \frac{12}{19}$ to get $\mathbf{q} = \begin{bmatrix} \frac{3}{19} \\ \frac{4}{19} \\ \frac{12}{19} \end{bmatrix}$.

4. (a) Prove by induction that $p_{12}^{(n)} = 0$: Already true for $n = 1$. If true for $n - 1$, we have $P^n = P^{n-1}P$, so $p_{12}^{(n)} = p_{11}^{(n-1)}p_{12} + p_{12}^{(n-1)}p_{22}$. But $p_{12} = p_{12}^{(n-1)} = 0$ so $p_{12}^{(n)} = 0 + 0 = 0$. Thus, no power of P can have all positive entries, so P is not regular.

(b) If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $P\mathbf{x} = \begin{bmatrix} \frac{1}{2}x_1 \\ \frac{1}{2}x_1 + x_2 \end{bmatrix}$,

$P^2\mathbf{x} = \begin{bmatrix} \frac{1}{4}x_1 \\ \frac{1}{4}x_1 + \frac{1}{2}x_1 + x_2 \end{bmatrix}$ etc. We use induction to show

$$P^n\mathbf{x} = \begin{bmatrix} \left(\frac{1}{2}\right)^n x_1 \\ \left(1-\left(\frac{1}{2}\right)^n\right)x_1 + x_2 \end{bmatrix}.$$

Already true for $n = 1, 2$. If true for $n - 1$, then $P^n = P(P^{n-1}\mathbf{x})$

$$= P\begin{bmatrix} \left(\frac{1}{2}\right)^{n-1} x_1 \\ \left(1-\left(\frac{1}{2}\right)^{n-1}\right)x_1 + x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} x_1 \\ \left(1-\left(\frac{1}{2}\right)^{n-1} + \left(\frac{1}{2}\right)^n\right)x_1 + x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{2}\right)^n x_1 \\ \left(1-\left(\frac{1}{2}\right)^n\right)x_1 + x_2 \end{bmatrix}.$$

Since $\lim_{n\to\infty} P^n\mathbf{x} = \begin{bmatrix} 0 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ if $\mathbf{x}$ is a state vector.

(c) The Theorem says that the entries of the steady state vector should be positive; they are not for $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

5. Let $\mathbf{q} = \begin{bmatrix} \frac{1}{k} \\ \frac{1}{k} \\ \vdots \\ \frac{1}{k} \end{bmatrix}$. Then

$$(P\mathbf{q})_i = \sum_{j=1}^{k} p_{ij}q_j = \sum_{j=1}^{k} \frac{1}{k}p_{ij} = \frac{1}{k}\sum_{j=1}^{k} p_{ij} = \frac{1}{k},$$

since the row sums of P are 1. Thus $(P\mathbf{q})_i = q_i$ for all i.

6. Since P has zeros entries, consider $P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$, so P is regular. Note that all the rows of P sum to 1. Since P is 3×3, Exercise 5 implies $\mathbf{q} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

7. Let $\mathbf{x} = [x_1 \quad x_2]^T$ be the state vector, with x_1 = probability that John is happy and x_2 = probability that John is sad. The transition matrix P will be $P = \begin{bmatrix} \frac{4}{5} & \frac{2}{3} \\ \frac{1}{5} & \frac{1}{3} \end{bmatrix}$ since the columns must sum to one. We find the steady state vector for P by solving $\begin{bmatrix} \frac{1}{5} & -\frac{2}{3} \\ -\frac{1}{5} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e., $\frac{1}{5}q_1 = \frac{2}{3}q_2$, so $\mathbf{q} = \begin{bmatrix} \frac{10}{3} \\ 1 \end{bmatrix}s$. Let $s = \frac{3}{13}$ and get $\mathbf{q} = \begin{bmatrix} \frac{10}{13} \\ \frac{3}{13} \end{bmatrix}$, so $\frac{10}{13}$ is the probability that John will be happy on a given day.

8. The state vector $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^T$ will represent the proportion of the population living in regions 1, 2, and 3, respectively. In the transition matrix, p_{ij} will represent the proportion of the people in region j who move to region i, yielding $P = \begin{bmatrix} .90 & .15 & .10 \\ .05 & .75 & .05 \\ .05 & .10 & .85 \end{bmatrix}$.

$$\begin{bmatrix} .10 & -.15 & -.10 \\ -.05 & .25 & -.05 \\ -.05 & -.10 & .15 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

First reduce to row echelon form $\begin{bmatrix} 1 & 0 & -\frac{13}{7} \\ 0 & 1 & -\frac{4}{7} \\ 0 & 0 & 0 \end{bmatrix}$, yielding $\mathbf{q} = \begin{bmatrix} \frac{13}{7} \\ \frac{4}{7} \\ 1 \end{bmatrix}s$. Set $s = \frac{7}{24}$ and get $q = \begin{bmatrix} \frac{13}{24} \\ \frac{4}{24} \\ \frac{7}{24} \end{bmatrix}$, i.e., in the long run $\frac{13}{24}$ $\left(\text{or } 54\frac{1}{6}\%\right)$ of the people reside in region 1, $\frac{4}{24}$ $\left(\text{or } 16\frac{2}{3}\%\right)$ in region 2, and $\frac{7}{24}$ $\left(\text{or } 29\frac{1}{6}\%\right)$ in region 3.

Section 10.6

Exercise Set 10.6

1. Note that the matrix has the same number of rows and columns as the graph has vertices, and that ones in the matrix correspond to arrows in the graph.

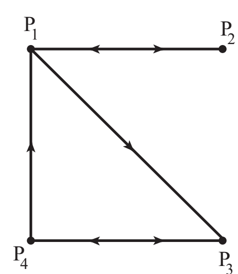
(a) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

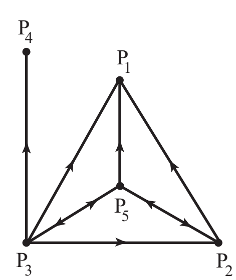
(c) $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

2. See the remark in problem 1; we obtain

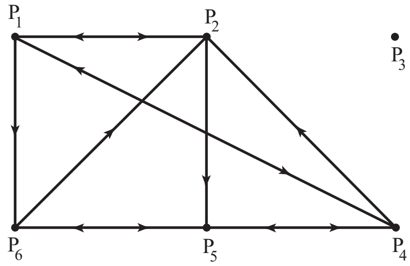
(a)



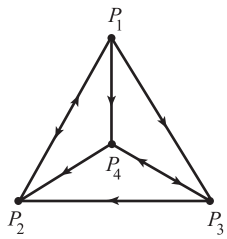
(b)



(c)



3. (a)



(b) $m_{12} = 1,$ so there is one 1-step connection from P_1 to P_2.

$$M^2 = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \text{ and}$$

$$M^3 = \begin{bmatrix} 2 & 3 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}.$$

So $m_{12}^{(2)} = 2$ and $m_{12}^{(3)} = 3$ meaning there are two 2-step and three 3-step connections from P_1 to P_2 by Theorem 10.6.1. These are:

1-step: $P_1 \to P_2$

2-step: $P_1 \to P_4 \to P_2$ and $P_1 \to P_3 \to P_2$

3-step: $P_1 \to P_2 \to P_1 \to P_2,$

$P_1 \to P_3 \to P_4 \to P_2,$ and

$P_1 \to P_4 \to P_3 \to P_2.$

(c) Since $m_{14} = 1,$ $m_{14}^{(2)} = 1$ and $m_{14}^{(3)} = 2,$ there are one 1-step, one 2-step and two 3-step connections from P_1 to P_4. These are:

1-step: $P_1 \to P_4$

2-step: $P_1 \to P_3 \to P_4$

3-step: $P_1 \to P_2 \to P_1 \to P_4$ and

$P_1 \to P_4 \to P_3 \to P_4.$

4. (a) $$M^T M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(b) The kth diagonal entry of $M^T M$ is $\sum_{i=1}^{5} m_{ik}{}^2,$ i.e., the sum of the squares of the entries in column k of M. These entries are 1 if family member i influences member k and 0 otherwise.

(c) The ij entry of $M^T M$ is the number of family members who influence both member i and member j.

5. (a) Note that to be contained in a clique, a vertex must have "two-way" connections with at least two other vertices. Thus, P_4 could not be in a clique, so $\{P_1, P_2, P_3\}$ is the only possible clique. Inspection shows that this is indeed a clique.

(b) Not only must a clique vertex have two-way connections to at least two other vertices, but the vertices to which it is connected must share a two-way connection. This consideration eliminates P_1 and $P_2,$ leaving $\{P_3, P_4, P_5\}$ as the only possible clique. Inspection shows that it is indeed a clique.

(c) The above considerations eliminate $P_1,$ P_3 and P_7 from being in a clique. Inspection shows that each of the sets $\{P_2, P_4, P_6\},$ $\{P_4, P_6, P_8\},$ $\{P_2, P_6, P_8\},$ $\{P_2, P_4, P_8\}$ and $\{P_4, P_5, P_6\}$ satisfy conditions (i) and (ii) in the definition of a clique. But note that P_8 can be added to the first set and we still satisfy the conditions. P_5 may not be added, so $\{P_2, P_4, P_6, P_8\}$ is a clique, containing all the other possibilities except $\{P_4, P_5, P_6\},$ which is also a clique.

6. (a) With the given M we get

$$S=\begin{bmatrix}0&1&0&1&0\\1&0&1&0&0\\0&1&0&0&1\\1&0&0&0&1\\0&0&1&1&0\end{bmatrix},$$

$$S^3=\begin{bmatrix}0&3&1&3&1\\3&0&3&1&1\\1&3&0&1&3\\3&1&1&0&3\\1&1&3&3&0\end{bmatrix}.$$

Since $s_{ii}^{(3)}=0$ for all i, there are no cliques in the graph represented by M.

(b) Here $S=\begin{bmatrix}0&1&0&1&0&0\\1&0&1&0&1&0\\0&1&0&1&0&1\\1&0&1&0&1&1\\0&1&0&1&0&0\\0&0&1&1&0&0\end{bmatrix},$

$$S^3=\begin{bmatrix}0&6&1&7&0&2\\6&0&7&1&6&3\\1&7&2&8&1&4\\7&1&8&2&7&5\\0&6&1&7&0&2\\2&3&4&5&2&2\end{bmatrix}.$$

The elements along the main diagonal tell us that only P_3, P_4, and P_6 are members of a clique. Since a clique contains at least three vertices, we must have $\{P_3, P_4, P_6\}$ as the only clique.

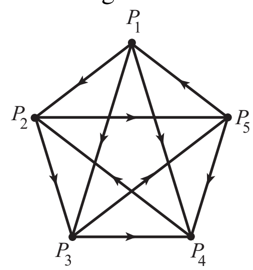
7. $M=\begin{bmatrix}0&0&1&1\\1&0&0&0\\0&1&0&1\\0&1&0&0\end{bmatrix}$

Then $M^2=\begin{bmatrix}0&2&0&1\\0&0&1&1\\1&1&0&0\\1&0&0&0\end{bmatrix}$ and

$$M+M^2=\begin{bmatrix}0&2&1&2\\1&0&1&1\\1&2&0&1\\1&1&0&0\end{bmatrix}.$$

By summing the rows of $M+M^2$, we get that the power of P_1 is 2 + 1 + 2 = 5, the power of P_2 is 3, of P_3 is 4, and of P_4 is 2.

8. Associating vertex P_1 with team A, P_2 with B, ..., P_5 with E, the game results yield the following dominance-directed graph:



which has vertex matrix

$$M=\begin{bmatrix}0&1&1&1&0\\0&0&1&0&1\\0&0&0&1&1\\0&1&0&0&0\\1&0&0&1&0\end{bmatrix}.$$

Then $M^2=\begin{bmatrix}0&1&1&1&2\\1&0&0&2&1\\1&1&0&1&0\\0&0&1&0&1\\0&2&1&1&0\end{bmatrix},$

$$M+M^2=\begin{bmatrix}0&2&2&2&2\\1&0&1&2&2\\1&1&0&2&1\\0&1&1&0&1\\1&2&1&2&0\end{bmatrix}.$$

Summing the rows, we get that the power of A is 8, of B is 6, of C is 5, of D is 3, and of E is 6. Thus ranking in decreasing order we get A in first place, B and E tie for second place, C in fourth place, and D last.

Section 10.7

Exercise Set 10.7

1. (a) From Equation (2), the expected payoff of the game is

$$\mathbf{p}A\mathbf{q}=\begin{bmatrix}\frac{1}{2}&0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}-4&6&-4&1\\5&-7&3&8\\-8&0&6&-2\end{bmatrix}\begin{bmatrix}\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}$$

$$=-\frac{5}{8}.$$

(b) If player *R* uses strategy $[p_1 \quad p_2 \quad p_3]$ against player *C*'s strategy $\begin{bmatrix}\frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\end{bmatrix}$ his payoff will be $\mathbf{p}A\mathbf{q} = \left(-\frac{1}{4}\right)\mathbf{p}_1 + \left(\frac{9}{4}\right)\mathbf{p}_2 - \mathbf{p}_3$. Since p_1, p_2, and p_3 are nonnegative and add up to 1, this is a weighted average of the numbers $-\frac{1}{4}$, $\frac{9}{4}$, and -1. Clearly this is the largest if $p_1 = p_3 = 0$ and $p_2 = 1$; that is, $\mathbf{p} = [0 \quad 1 \quad 0]$.

(c) As in (b), if player *C* uses $[q_1 \quad q_2 \quad q_3 \quad q_4]^T$ against $\left[\frac{1}{2} \quad 0 \quad \frac{1}{2}\right]$, we get $\mathbf{p}A\mathbf{q} = -6q_1 + 3q_2 + q_3 - \frac{1}{2}q_4$. Clearly this is minimized over all strategies by setting $q_1 = 1$ and $q_2 = q_3 = q_4 = 0$. That is $\mathbf{q} = [1 \quad 0 \quad 0 \quad 0]^T$.

2. As per the hint, we will construct a 3×3 matrix with two saddle points, say $a_{11} = a_{33} = 1$. Such a matrix is $A = \begin{bmatrix}1 & 2 & 1\\ 0 & 7 & 0\\ 1 & 2 & 1\end{bmatrix}$. Note that $a_{13} = a_{31} = 1$ are also saddle points.

3. (a) Calling the matrix *A*, we see a_{22} is a saddle point, so the optimal strategies are pure, namely: $\mathbf{p}^* = [0 \quad 1]$, $\mathbf{q}^* = \begin{bmatrix}0\\ 1\end{bmatrix}$, the value of the game is $v = a_{22} = 3$.

(b) As in (a), a_{21} is a saddle point, so optimal strategies are $\mathbf{p}^* = [0 \quad 1 \quad 0]$, $\mathbf{q}^* = \begin{bmatrix}1\\ 0\end{bmatrix}$, the value of the game is $v = a_{21} = 2$.

(c) Here, a_{32} is a saddle point, so optimal strategies are $\mathbf{p}^* = [0 \quad 0 \quad 1]$, $\mathbf{q}^* = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$ and $v = a_{32} = 2$.

(d) Here, a_{21} is a saddle point, so $\mathbf{p}^* = [0 \quad 1 \quad 0 \quad 0]$, $\mathbf{q}^* = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$ and $v = a_{21} = -2$.

4. (a) Calling the matrix *A*, the formulas of Theorem 10.7.2 yield $\mathbf{p}^* = \left[\frac{5}{8} \quad \frac{3}{8}\right]$, $\mathbf{q}^* = \begin{bmatrix}\frac{1}{8}\\ \frac{7}{8}\end{bmatrix}$, $v = \frac{27}{8}$ (*A* has no saddle points).

(b) As in (a), $\mathbf{p}^* = \left[\frac{40}{60} \quad \frac{20}{60}\right] = \left[\frac{2}{3} \quad \frac{1}{3}\right]$, $\mathbf{q}^* = \begin{bmatrix}\frac{10}{60}\\ \frac{50}{60}\end{bmatrix} = \begin{bmatrix}\frac{1}{6}\\ \frac{5}{6}\end{bmatrix}$, $v = \frac{1400}{60} = \frac{70}{3}$ (Again, *A* has no saddle points).

(c) For this matrix, a_{11} is a saddle point, so $\mathbf{p}^* = [1 \quad 0]$, $\mathbf{q}^* = \begin{bmatrix}1\\ 0\end{bmatrix}$, and $v = a_{11} = 3$.

(d) This matrix has no saddle points, so, as in (a), $\mathbf{p}^* = \left[\frac{-3}{-5} \quad \frac{-2}{-5}\right] = \left[\frac{3}{5} \quad \frac{2}{5}\right]$, $\mathbf{q}^* = \begin{bmatrix}\frac{-3}{-5}\\ \frac{-2}{-5}\end{bmatrix} = \begin{bmatrix}\frac{3}{5}\\ \frac{2}{5}\end{bmatrix}$, and $v = \frac{-19}{-5} = \frac{19}{5}$.

(e) Again, *A* has no saddle points, so as in (a), $\mathbf{p}^* = \left[\frac{3}{13} \quad \frac{10}{13}\right]$, $\mathbf{q}^* = \begin{bmatrix}\frac{1}{13}\\ \frac{12}{13}\end{bmatrix}$, and $v = \frac{-29}{13}$.

5. Let
a_{11} = payoff to *R* if the black ace and black two are played = 3.
a_{12} = payoff to *R* if the black ace and red three are played = −4.
a_{21} = payoff to *R* if the red four and black two are played = −6.
a_{22} = payoff to *R* if the red four and red three

are played = 7.
So, the payoff matrix for the game is
$A = \begin{bmatrix} 3 & -4 \\ -6 & 7 \end{bmatrix}.$
A has no saddle points, so from Theorem 10.7.2,
$\mathbf{p}^* = \begin{bmatrix} \frac{13}{20} & \frac{7}{20} \end{bmatrix}$, $\mathbf{q}^* = \begin{bmatrix} \frac{11}{20} \\ \frac{9}{20} \end{bmatrix}$; that is, player R should play the black ace 65 percent of the time, and player C should play the black two 55 percent of the time. The value of the game is $-\frac{3}{20}$, that is, player C can expect to collect on the average 15 cents per game.

Section 10.8

Exercise Set 10.8

1. (a) Calling the given matrix E, we need to solve
$$(I-E)\mathbf{p} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{2} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
This yields $\frac{1}{2}p_1 = \frac{1}{3}p_2$, that is, $\mathbf{p} = s\begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}$.

Set $s = 2$ and get $\mathbf{p} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(b) As in (a), solve
$$(I-E)\mathbf{p} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{3} & 1 & -\frac{1}{2} \\ -\frac{1}{6} & -1 & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
In row-echelon form, this reduces to
$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
Solutions of this system have the form
$\mathbf{p} = s\begin{bmatrix} 1 \\ \frac{5}{6} \\ 1 \end{bmatrix}$. Set $s = 6$ and get $\mathbf{p} = \begin{bmatrix} 6 \\ 5 \\ 6 \end{bmatrix}$.

(c) As in (a), solve
$$(I-E)\mathbf{p} = \begin{bmatrix} .65 & -.50 & -.30 \\ -.25 & .80 & -.30 \\ -.40 & -.30 & .60 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$
which reduces to $\begin{bmatrix} 1 & 0 & -\frac{78}{79} \\ 0 & 1 & -\frac{54}{79} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Solutions are of the form $\mathbf{p} = \begin{bmatrix} \frac{78}{79} \\ \frac{54}{79} \\ 1 \end{bmatrix}$. Let

$s = 79$ to obtain $\mathbf{p} = \begin{bmatrix} 78 \\ 54 \\ 79 \end{bmatrix}$.

2. (a) By Corollary 10.8.4, this matrix is productive, since each of its row sums is .9.

(b) By Corollary 10.8.5, this matrix is productive, since each of its column sums is less than one.

(c) Try $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Then $C\mathbf{x} = \begin{bmatrix} 1.9 \\ .9 \\ .9 \end{bmatrix}$, i.e., $\mathbf{x} > C\mathbf{x}$, so this matrix is productive by Theorem 10.8.3.

3. Theorem 10.8.2 says there will be one linearly independent price vector for the matrix E if some positive power of E is positive. Since E is not positive, try E^2.
$$E^2 = \begin{bmatrix} .2 & .34 & .1 \\ .2 & .54 & .6 \\ .6 & .12 & .3 \end{bmatrix} > 0$$

4. The exchange matrix for this arrangement (using A, B, and C in that order) is $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$.

For equilibrium, we must solve $(I-E)\mathbf{p} = 0$.

That is $\begin{bmatrix} \frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Row reduction yields solutions of the form
$\mathbf{p} = \begin{bmatrix} \frac{18}{16} \\ \frac{15}{16} \\ 1 \end{bmatrix}s$. Set $s = \frac{1600}{15}$ and obtain

$\mathbf{p} = \begin{bmatrix} 120 \\ 100 \\ 106.67 \end{bmatrix}$; i.e., the price of tomatoes was $120, corn was $100, and lettuce was $106.67.

5. Taking the CE, EE, and ME in that order, we form the consumption matrix C, where c_{ij} = the amount (per consulting dollar) of the i-th engineer's services purchased by the j-th engineer. Thus, $C = \begin{bmatrix} 0 & .2 & .3 \\ .1 & 0 & .4 \\ .3 & .4 & 0 \end{bmatrix}$.

We want to solve $(I - C)\mathbf{x} = \mathbf{d}$, where $\mathbf{d}$ is the demand vector, i.e.

$$\begin{bmatrix} 1 & -.2 & -.3 \\ -.1 & 1 & -.4 \\ -.3 & -.4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 500 \\ 700 \\ 600 \end{bmatrix}.$$

In row-echelon form this reduces to

$$\begin{bmatrix} 1 & -.2 & -.3 \\ 0 & 1 & -.43877 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 500 \\ 765.31 \\ 1556.19 \end{bmatrix}.$$

Back-substitution yields the solution

$$\mathbf{x} = \begin{bmatrix} 1256.48 \\ 1448.12 \\ 1556.19 \end{bmatrix}.$$

The CE received $1256, the EE received $1448, and the ME received $1556.

6. (a) The solution of the system $(I - C)\mathbf{x} = \mathbf{d}$ is $\mathbf{x} = (I - C)^{-1}\mathbf{d}$. The effect of increasing the demand d_i for the ith industry by one unit is the same as adding $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ to $\mathbf{d}$ where the 1 is in the ith row. The new solution is

$$(I - C)^{-1}\left(\mathbf{d} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = \mathbf{x} + (I - C)^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which has the effect of adding the ith column of $(I - C)^{-1}$ to the original solution.

(b) The increase in value is the second column of $(I - C)^{-1}$, $\frac{1}{503}\begin{bmatrix} 542 \\ 690 \\ 170 \end{bmatrix}$. Thus the value of the coal-mining operation must increase by $\frac{542}{503}$.

7. The i-th column sum of E is $\sum_{j=1}^{n} e_{ji}$, and the elements of the i-th column of $I - E$ are the negatives of the elements of E, except for the ii-th, which is $1 - e_{ii}$. So, the i-th column sum of $I - E$ is $1 - \sum_{j=1}^{n} e_{ji} = 1 - 1 = 0$. Now, $(I - E)^T$ has zero row sums, so the vector $\mathbf{x} = [1 \quad 1 \quad \cdots \quad 1]^T$ solves $(I - E)^T\mathbf{x} = 0$. This implies $\det(I - E)^T = 0$. But $\det(I - E)^T = \det(I - E)$, so $(I - E)\mathbf{p} = 0$ must have nontrivial (i.e., nonzero) solutions.
The CE received $1256, the EE received $1448, and the ME received $1556.

8. Let C be a consumption matrix whose column sums are less than one; then the row sums of C^T are less than one. By Corollary 10.8.4, C^T is productive so $(I - C^T)^{-1} \geq 0$. But

$$\begin{aligned}(I - C)^{-1} &= (((I - C)^T))^{-1})^T \\ &= ((I - C^T)^{-1})^T \\ &\geq 0.\end{aligned}$$

Thus, C is productive.

Section 10.9

Exercise Set 10.9

1. Using Equation (18), we calculate

$$Yld_2 = \frac{30s}{2} = 15s$$

$$Yld_3 = \frac{50s}{2 + \frac{3}{2}} = \frac{100s}{7}.$$

So all the trees in the second class should be harvested for an optimal yield (since s = 1000) of $15,000.

2. From the solution to Example 1, we see that for the fifth class to be harvested in the optimal case we must have $\dfrac{p_5 s}{\left(.28^{-1}+.31^{-1}+.25^{-1}+.23^{-1}\right)} > 14.7s$, yielding $p_5 > \$222.63$.

3. Assume $p_2 = 1$, then $Yld_2 = \dfrac{s}{(.28)^{-1}} = .28s$. Thus, for all the yields to be the same we must have

$$\frac{p_3 s}{(.28^{-1}+.31^{-1})} = .28s$$
$$\frac{p_4 s}{(.28^{-1}+.31^{-1}+.25^{-1})} = .28s$$
$$\frac{p_5 s}{(.28^{-1}+.31^{-1}+.25^{-1}+.23^{-1})} = .28s$$
$$\frac{p_6 s}{(.28^{-1}+.31^{-1}+.25^{-1}+.23^{-1}+.37^{-1})} = .28s$$

Solving these successively yields $p_3 = 1.90$, $p_4 = 3.02$, $p_5 = 4.24$ and $p_6 = 5.00$. Thus the ratio $p_2 : p_3 : p_4 : p_5 : p_6 = 1 : 1.90 : 3.02 : 4.24 : 5.00$.

5. Since **y** is the harvest vector, $N = \sum_{i=1}^{n} y_i$ is the number of trees removed from the forest. Then Equation (7) and the first of Equations (8) yield $N = g_1 x_1$, and from Equation (17) we obtain

$$N = \frac{g_1 s}{1+\frac{g_1}{g_2}+\cdots+\frac{g_1}{g_{k-1}}} = \frac{s}{\frac{1}{g_1}+\cdots+\frac{1}{g_{k-1}}}.$$

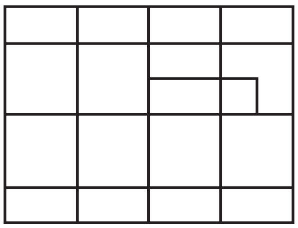
6. Set $g_1 = \cdots = g_{n-1} = g$, and $p_2 = 1$. Then from Equation (18), $Yld_2 = \dfrac{p_2 s}{\frac{1}{g_1}} = gs$. Since we want all of the Yld_k's to be the same, we need to solve $Yld_k = \dfrac{p_k s}{(k-1)\frac{1}{g}} = gs$ for p_k for $3 \le k \le n$. Thus $p_k = k-1$. So the ratio $p_2 : p_3 : p_4 : \cdots : p_n = 1 : 2 : 3 : \cdots : (n-1)$.

Section 10.10

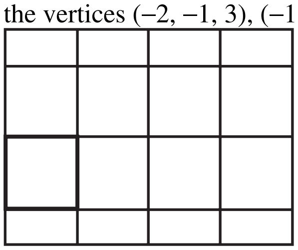
Exercise Set 10.10

1. (a) Using the coordinates of the points as the columns of a matrix we obtain $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) The scaling is accomplished by multiplication of the coordinate matrix on the left by $\begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, resulting in the matrix $\begin{bmatrix} 0 & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which represents the vertices (0, 0, 0), $\left(\frac{3}{2}, 0, 0\right)$, $\left(\frac{3}{2}, \frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}, 0\right)$ as shown below.



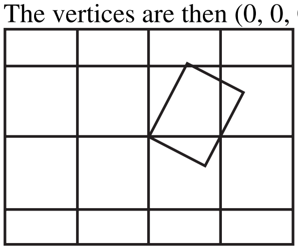
(c) Adding the matrix $\begin{bmatrix} -2 & -2 & -2 & -2 \\ -1 & -1 & -1 & -1 \\ 3 & 3 & 3 & 3 \end{bmatrix}$ to the original matrix yields $\begin{bmatrix} -2 & -1 & -1 & -2 \\ -1 & -1 & 0 & 0 \\ 3 & 3 & 3 & 3 \end{bmatrix}$, which represents the vertices (−2, −1, 3), (−1, −1, 3), (−1, 0, 3), and (−2, 0, 3) as shown below.



(d) Multiplying by the matrix $\begin{bmatrix} \cos(-30°) & -\sin(-30°) & 0 \\ \sin(-30°) & \cos(-30°) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we obtain

$$\begin{bmatrix} 0 & \cos(-30°) & \cos(-30°)-\sin(-30°) & -\sin(-30°) \\ 0 & \sin(-30°) & \cos(-30°)+\sin(-30°) & \cos(-30°) \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & .866 & 1.366 & .500 \\ 0 & -.500 & .366 & .866 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The vertices are then (0, 0, 0), (.866, −.500, 0), (1.366, .366, 0), and (.500, .866, 0) as shown:



2. (a) Simply perform the matrix multiplication $\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} x_i + \frac{1}{2} y_i \\ y_i \\ z_i \end{bmatrix}$.

(b) We multiply

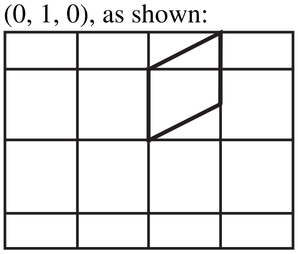
$$\begin{bmatrix}1 & \frac{1}{2} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 1 & \frac{3}{2} & \frac{1}{2}\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

yielding the vertices (0, 0, 0), (1, 0, 0), $\left(\frac{3}{2}, 1, 0\right)$, and $\left(\frac{1}{2}, 1, 0\right)$.

(c) Obtain the vertices via

$$\begin{bmatrix}1 & 0 & 0\\ .6 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 1 & 1 & 0\\ 0 & .6 & 1.6 & 1\\ 0 & 0 & 0 & 0\end{bmatrix},$$

yielding (0, 0, 0), (1, .6, 0), (1, 1.6, 0), and (0, 1, 0), as shown:

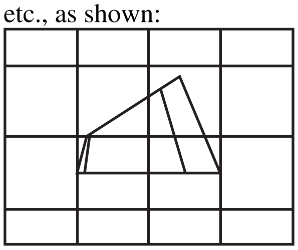


3. (a) This transformation looks like scaling by the factors 1, −1, 1, respectively and indeed its matrix is $\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$.

(b) For this reflection we want to transform (x_i, y_i, z_i) to $(-x_i, y_i, z_i)$ with the matrix

$$\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Negating the x-coordinates of the 12 points in view 1 yields the 12 points (−1.000, −.800, .000), (−.500, −.800, −.866), etc., as shown:



(c) Here we want to negate the z-coordinates, with the matrix $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$.

This does not change View 1.

4. (a) The formulas for scaling, translation, and rotation yield the matrices

$$M_1=\begin{bmatrix}\frac{1}{2} & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & \frac{1}{3}\end{bmatrix},$$

$$M_2=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2}\\ 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix},$$

$$M_3=\begin{bmatrix}1 & 1 & 0\\ 0 & \cos 20° & -\sin 20°\\ 0 & \sin 20° & \cos 20°\end{bmatrix},$$

$$M_4=\begin{bmatrix}\cos(-45°) & 0 & \sin(-45)°\\ 0 & 1 & 0\\ -\sin(-45°) & 0 & \cos(-45°)\end{bmatrix}, \text{ and}$$

$$M_5=\begin{bmatrix}\cos 90° & -\sin 90° & 0\\ \sin 90° & \cos 90° & 0\\ 0 & 0 & 1\end{bmatrix}.$$

(b) Clearly $P' = M_5M_4M_3(M_1P + M_2)$.

5. (a) As in 4(a), $M_1=\begin{bmatrix}.3 & 0 & 0\\ 0 & .5 & 0\\ 0 & 0 & 1\end{bmatrix}$,

$$M_2=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos 45° & -\sin 45°\\ 0 & \sin 45° & \cos 45°\end{bmatrix},$$

$$M_3=\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix},$$

$$M_4=\begin{bmatrix}\cos 35° & 0 & \sin 35°\\ 0 & 1 & 0\\ -\sin 35° & 0 & \cos 35°\end{bmatrix},$$

$$M_5=\begin{bmatrix}\cos(-45°) & -\sin(-45°) & 0\\ \sin(-45°) & \cos(-45°) & 0\\ 0 & 0 & 1\end{bmatrix},$$

$$M_6=\begin{bmatrix}0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ 1 & 1 & 1 & \cdots & 1\end{bmatrix}, \text{ and}$$

$$M_7=\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

(b) As in 4(b), $P' = M_7(M_6 + M_5M_4(M_3 + M_2M_1P))$.

6. Using the hing given, we have

$$R_1 = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$R_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$R_4 = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$R_5 = \begin{bmatrix} \cos(-\beta) & 0 & \sin(-\beta) \\ 0 & 1 & 0 \\ -\sin(-\beta) & 0 & \cos(-\beta) \end{bmatrix}.$$

7. (a) We rewrite the formula for v_i' as

$$v_i' = \begin{bmatrix} 1\cdot x_i + x_0\cdot 1 \\ 1\cdot y_i + y_0\cdot 1 \\ 1\cdot z_i + z_0\cdot 1 \\ 1\cdot 1 \end{bmatrix}. \text{ So}$$

$$v_i' = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}.$$

(b) We want to translate x_i by –5, y_i by +9, z_i by –3, so $x_0 = -5$, $y_0 = 9$, $z_0 = -3$.

The matrix is $\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

8. This can be done most easily performing the multiplication RR^T and showing that this is I. For example, for the rotation matrix about the x-axis we obtain

$$RR^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Section 10.11

Exercise Set 10.11

1. (a) The discrete mean value property yields the four equations

$$t_1 = \frac{1}{4}(t_2 + t_3)$$

$$t_2 = \frac{1}{4}(t_1 + t_4 + 1 + 1)$$

$$t_3 = \frac{1}{4}(t_1 + t_4)$$

$$t_4 = \frac{1}{4}(t_2 + t_3 + 1 + 1).$$

Translated into matrix notation, this becomes

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}.$$

(b) To solve the system in part (a), we solve $(I - M)\mathbf{t} = \mathbf{b}$ for $\mathbf{t}$:

$$\begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 1 & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & 1 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}.$$

In row-echelon form, this is

$$\begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & -15 & 4 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{8} \\ \frac{3}{4} \end{bmatrix}.$$

Back substitution yields the result $\mathbf{t} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}.$

(c) $\mathbf{t}^{(1)} = M\mathbf{t}^{(0)} + \mathbf{b}$

$$= \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

$$\mathbf{t}^{(2)} = M\mathbf{t}^{(1)} + \mathbf{b} = \begin{bmatrix} \frac{1}{8} \\ \frac{5}{8} \\ \frac{1}{8} \\ \frac{5}{8} \end{bmatrix},$$

$$\mathbf{t}^{(3)} = M\mathbf{t}^{(2)} + \mathbf{b} = \begin{bmatrix} \frac{3}{16} \\ \frac{11}{16} \\ \frac{3}{16} \\ \frac{11}{16} \end{bmatrix}$$

$$\mathbf{t}^{(4)} = M\mathbf{t}^{(3)} + \mathbf{b} = \begin{bmatrix} \frac{7}{32} \\ \frac{23}{32} \\ \frac{7}{32} \\ \frac{23}{32} \end{bmatrix},$$

$$\mathbf{t}^{(5)} = M\mathbf{t}^{(4)} + \mathbf{b} = \begin{bmatrix} \frac{15}{64} \\ \frac{47}{64} \\ \frac{15}{64} \\ \frac{47}{64} \end{bmatrix}$$

$$\mathbf{t}^{(5)} - \mathbf{t} = \begin{bmatrix} \frac{15}{64} \\ \frac{47}{64} \\ \frac{15}{64} \\ \frac{47}{64} \end{bmatrix} - \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{64} \\ -\frac{1}{64} \\ -\frac{1}{64} \\ -\frac{1}{64} \end{bmatrix}$$

(d) Using
percentage error

$$= \frac{\text{computed value} - \text{actual value}}{\text{actual value}} \times 100\%$$

we have that the percentage error for t_1 and t_3 was $\frac{-.0371}{.2871} \times 100\% = -12.9\%$, and for t_2 and t_4 was $\frac{.0371}{.7129} \times 100\% = 5.2\%$.

2. The average value of the temperature on the circle is $\frac{1}{2\pi r}\int_{-\pi}^{\pi} f(\theta) r\, d\theta$, where r is the radius of the circle and $f(\theta)$ is the temperature at the point of the circumference where the radius to that point makes the angle θ with the horizontal. Clearly $f(\theta) = 1$ for $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$ and is zero otherwise. Consequently, the value of the integral above (which equals the temperature at the center of the circle) is $\frac{1}{2}$.

3. As in 1(c), but using M and $\mathbf{b}$ as in the problem statement, we obtain

$$\mathbf{t}^{(1)} = M\mathbf{t}^{(0)} + \mathbf{b}$$
$$= \begin{bmatrix} \frac{3}{4} & \frac{5}{4} & \frac{1}{2} & \frac{5}{4} & 1 & \frac{1}{2} & \frac{5}{4} & 1 & \frac{3}{4} \end{bmatrix}^T$$
$$\mathbf{t}^{(2)} = M\mathbf{t}^{(1)} + \mathbf{b}$$
$$= \begin{bmatrix} \frac{13}{16} & \frac{9}{8} & \frac{9}{16} & \frac{11}{8} & \frac{13}{16} & \frac{7}{16} & \frac{21}{16} & 1 & \frac{15}{8} \end{bmatrix}^T.$$

Section 10.12

Exercise Set 10.12

1. (c) The linear system

$$x_{31}^* = \frac{1}{20}[28 + x_{31}^* - x_{32}^*]$$
$$x_{32}^* = \frac{1}{20}[24 + 3x_{31}^* - 3x_{32}^*]$$

can be rewritten as

$$19x_{31}^* + x_{32}^* = 28$$
$$-3x_{31}^* + 23x_{32}^* = 24,$$

which has the solution

$$x_{31}^* = \frac{31}{22}$$
$$x_{32}^* = \frac{27}{22}.$$

2. (a) Setting $\mathbf{x}_0^{(1)} = \left(x_{01}^{(1)}, x_{02}^{(1)}\right) = \left(x_{31}^{(0)}, x_{32}^{(0)}\right) = (0, 0)$, and using part (b) of Exercise 1, we have

$x_{31}^{(1)} = \frac{1}{20}[28] = 1.40000$

$x_{32}^{(1)} = \frac{1}{20}[24] = 1.20000$

$x_{31}^{(2)} = \frac{1}{20}[28 + 1.4 - 1.2] = 1.41000$

$x_{32}^{(2)} = \frac{1}{20}[24 + 3(1.4) - 3(1.2)] = 1.23000$

$x_{31}^{(3)} = \frac{1}{20}[28 + 1.41 - 1.23] = 1.40900$

$x_{32}^{(3)} = \frac{1}{20}[24 + 3(1.41) - 3(1.23)] = 1.22700$

$x_{31}^{(4)} = \frac{1}{20}[28 + 1.409 - 1.227] = 1.40910$

$x_{32}^{(4)} = \frac{1}{20}[24 + 3(1.409) - 3(1.227)]$
$= 1.22730$

$x_{31}^{(5)} = \frac{1}{20}[28 + 1.4091 - 1.2273] = 1.40909$

$x_{32}^{(5)} = \frac{1}{20}[24 + 3(1.4091) - 3(1.2273)]$
$= 1.22727$

$x_{31}^{(6)} = \frac{1}{20}[28 + 1.40909 - 1.22727]$
$= 1.40909$

$x_{32}^{(6)} = \frac{1}{20}[24 + 3(1.40909) - 3(1.22727)]$
$= 1.22727.$

(b) $\mathbf{x}_0^{(1)} = (1, 1) = \left(x_{31}^{(0)}, x_{32}^{(0)}\right)$

$x_{31}^{(1)} = \frac{1}{20}[28 + 1 - 1] = 1.4$

$x_{32}^{(2)} = \frac{1}{20}[24 + 3(1) - 3(1)] = 1.2$

Since $\mathbf{x}_3^{(1)}$ in this part is the same as $\mathbf{x}_3^{(1)}$ in part (a), we will get $\mathbf{x}_3^{(2)}$ as in part (a) and therefore $\mathbf{x}_3^{(3)}, \ldots, \mathbf{x}_3^{(6)}$ will also be the same as in part (a).

(c) $\mathbf{x}_0^{(1)} = (148, -15) = \left(x_{31}^{(0)}, x_{32}^{(0)}\right)$

$x_{31}^{(1)} = \frac{1}{20}[28 + 148 - (-15)] = 9.55000$

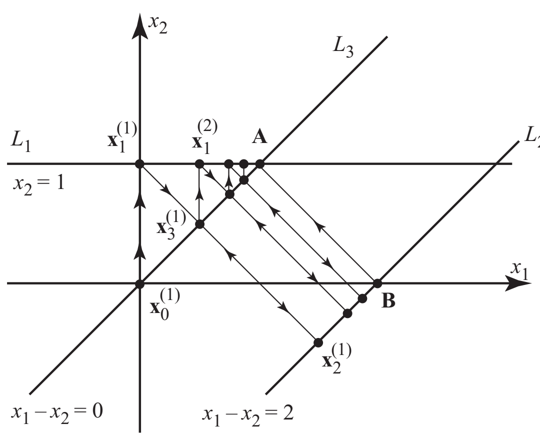
$x_{32}^{(1)} = \frac{1}{20}[24 + 3(148) - 3(-15)] = 25.65000$

$x_{31}^{(2)} = \frac{1}{20}[28 + 9.55 - 25.65] = 0.59500$

$$\begin{aligned}
x_{32}^{(2)} &= \frac{1}{20}[24+3(9.55)-3(25.65)] \\
&= -1.21500 \\
x_{31}^{(3)} &= \frac{1}{20}[28+0.595+1.215] \\
&= 1.49050 \\
x_{32}^{(3)} &= \frac{1}{20}[24+3(0.595)+3(1.215)] \\
&= 1.47150 \\
x_{31}^{(4)} &= \frac{1}{20}[28+1.4905-1.4715] = 1.40095 \\
x_{32}^{(4)} &= \frac{1}{20}[24+3(1.4905)-3(1.4715)] \\
&= 1.20285 \\
x_{31}^{(5)} &= \frac{1}{20}[28+1.40095-1.20285] \\
&= 1.40991 \\
x_{32}^{(5)} &= \frac{1}{20}[24+3(1.40095)-3(1.20285)] \\
&= 1.22972 \\
x_{31}^{(6)} &= \frac{1}{20}[28+1.40991-1.22972] = 1.40901 \\
x_{32}^{(6)} &= \frac{1}{20}[24+3(1.40991)-3(1.22972)] \\
&= 1.22703
\end{aligned}$$

4. Referring to the figure below and starting with $\mathbf{x}_0^{(1)} = (0, 0)$:

$\mathbf{x}_0^{(1)}$ is projected to $\mathbf{x}_1^{(1)}$ on L_1, $\mathbf{x}_1^{(1)}$ is projected to $\mathbf{x}_2^{(1)}$ on L_2, $\mathbf{x}_2^{(1)}$ is projected to $\mathbf{x}_3^{(1)}$ on L_3, and so on.



As seen from the graph the points of the limit cycle are $\mathbf{x}_1^* = \mathbf{A}$, $\mathbf{x}_2^* = \mathbf{B}$, $\mathbf{x}_3^* = \mathbf{A}$.

Since $\mathbf{x}_1^*$ is the point of intersection of L_1 and L_3 it follows on solving the system

$$\begin{aligned}
x_{12}^* &= 1 \\
x_{11}^* - x_{12}^* &= 0
\end{aligned}$$

that $\mathbf{x}_1^* = (1, 1)$. Since $\mathbf{x}_2^* = (x_{21}^*, x_{22}^*)$ is on L_2, it follows that $x_{21}^* - x_{22}^* = 2$. Now $\overline{\mathbf{x}_1^*\mathbf{x}_2^*}$ is perpendicular to L_2, therefore $\left(\dfrac{x_{22}^*-1}{x_{21}^*-1}\right)(1)=-1$ so we have $x_{22}^*-1=1-x_{21}^*$ or $x_{21}^*+x_{22}^*=2.$

Solving the system

$x_{21}^*-x_{22}^*=2$
$x_{21}^*+x_{22}^*=2$

gives $x_{21}^*=2$ and $x_{22}^*=0.$ Thus the points on the limit cycle are $\mathbf{x}_1^*=(1, 1),$ $\mathbf{x}_2^*=(2, 0),$ $\mathbf{x}_3^*=(1, 1).$

7. Let us choose units so that each pixel is one unit wide. Then a_{ij} = length of the center line of the *i*-th beam that lies in the *j*-th pixel.

If the *i*-th beam crosses the *j*-th pixel squarely, it follows that $a_{ij}=1.$ From Fig. 10.12.11 in the text, it is then clear that

$a_{17}=a_{18}=a_{19}=1$
$a_{24}=a_{25}=a_{26}=1$
$a_{31}=a_{32}=a_{33}=1$
$a_{73}=a_{76}=a_{79}=1$
$a_{82}=a_{85}=a_{88}=1$
$a_{91}=a_{94}=a_{97}=1$

since beams 1, 2, 3, 7, 8, and 9 cross the pixels squarely. Next, the centerlines of beams 5 and 11 lie along the diagonals of pixels 3, 5, 7 and 1, 5, 9, respectively. Since these diagonals have length $\sqrt{2},$ we have

$a_{53}=a_{55}=a_{57}=\sqrt{2}=1.41421$
$a_{11,\,1}=a_{11,\,5}=a_{11,\,9}=\sqrt{2}=1.41421.$

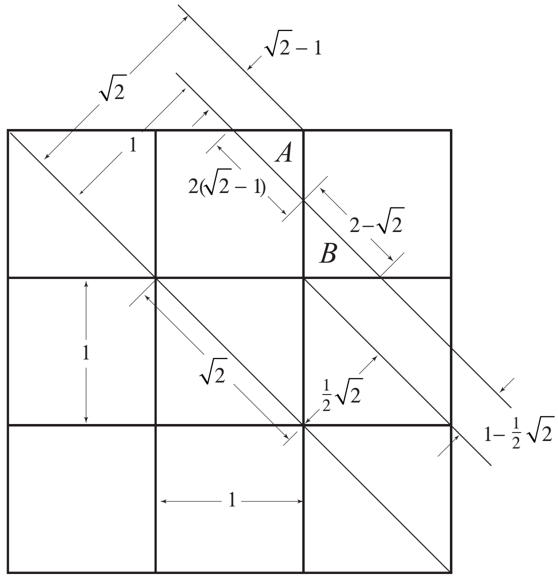
In the following diagram, the hypotenuse of triangle *A* is the portion of the centerline of the 10th beam that lies in the 2nd pixel. The length of this hypotenuse is twice the height of triangle *A*, which in turn is $\sqrt{2}-1.$ Thus,

$a_{10,\,2}=2\left(\sqrt{2}-1\right)=.82843.$

By symmetry we also have

$a_{10,\,2}=a_{10,\,6}=a_{12,\,4}=a_{12,\,8}=a_{62}=a_{64}=a_{46}=a_{48}=.82843.$

Also from the diagram, we see that the hypotenuse of triangle *B* is the portion of the centerline of the 10th beam that lies in the 3rd pixel. Thus, $a_{10,\,3}=2-\sqrt{2}=.58579.$



By symmetry we have $a_{10,\,3} = a_{12,\,7} = a_{61} = a_{49} = .58579.$

The remaining a_{ij} 's are all zero, and so the 12 beam equations (4) are

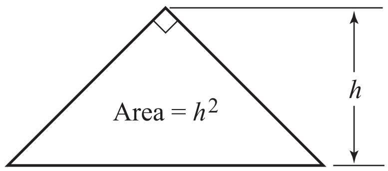
$$
\begin{aligned}
x_7 + x_8 + x_9 &= 13.00 \\
x_4 + x_5 + x_6 &= 15.00 \\
x_1 + x_2 + x_3 &= 8.00 \\
.82843(x_6 + x_8) + .58579x_9 &= 14.79 \\
1.41421(x_3 + x_5 + x_7) &= 14.31 \\
.82843(x_2 + x_4) + .58579x_1 &= 3.81 \\
x_3 + x_6 + x_9 &= 18.00 \\
x_2 + x_5 + x_8 &= 12.00 \\
x_1 + x_4 + x_7 &= 6.00 \\
.82843(x_2 + x_6) + .58579x_3 &= 10.51 \\
1.41421(x_1 + x_5 + x_9) &= 16.13 \\
.82843(x_4 + x_8) + .58579x_7 &= 7.04
\end{aligned}
$$

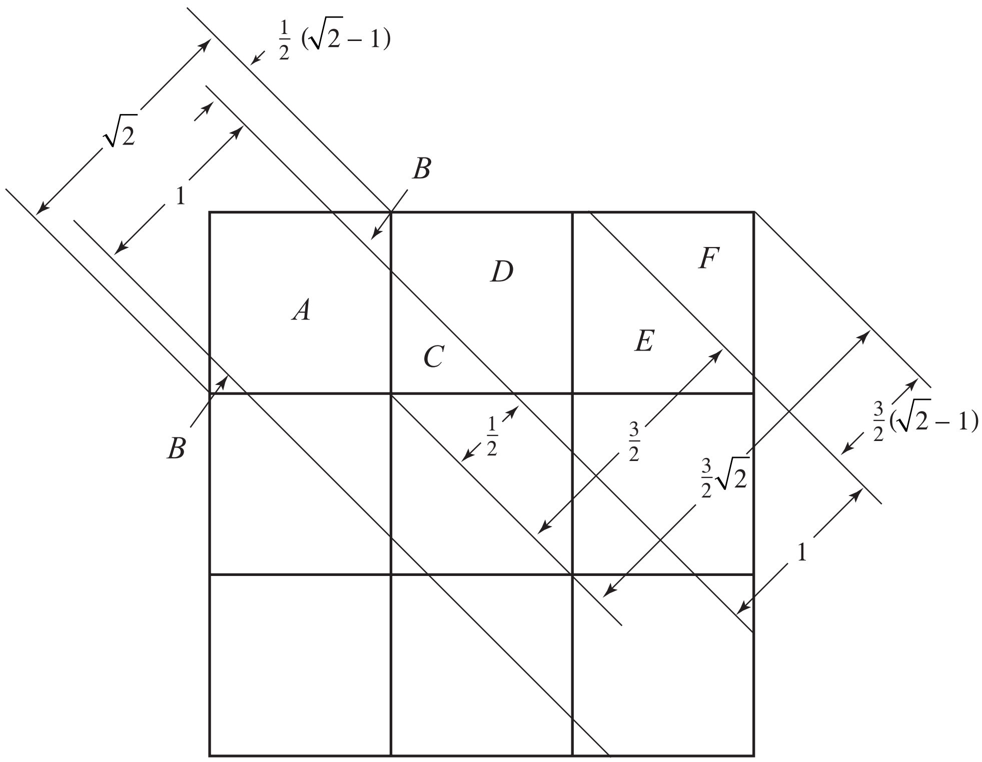
8. Let us choose units so that each pixel is one unit wide. Then a_{ij} = area of the *i*-th beam that lies in the *j*-th pixel. Since the width of each beam is also one unit it follows that $a_{ij} = 1$ if the *i*-th beam crosses the *j*-th pixel squarely. From Fig. 10.12.11 in the text, it is then clear that

$a_{17} = a_{18} = a_{19} = 1$
$a_{24} = a_{25} = a_{26} = 1$
$a_{31} = a_{32} = a_{33} = 1$
$a_{73} = a_{76} = a_{79} = 1$
$a_{82} = a_{85} = a_{88} = 1$
$a_{91} = a_{94} = a_{97} = 1$

since beams 1, 2, 3, 7, 8, and 9 cross the pixels squarely.

For the remaining a_{ij} 's, first observe from the figure that an isosceles right triangle of height *h*, as indicated, has area h^2. From the diagram of the nine pixels, we then have





$$\begin{aligned}\text{Area of triangle } B &= \left[\frac{1}{2}\left(\sqrt{2}-1\right)\right]^2\\ &= \frac{1}{4}\left(3-2\sqrt{2}\right)\\ &= 0.4289\end{aligned}$$

$$\text{Area of triangle } C = \left[\frac{1}{2}\right]^2 = \frac{1}{4} = .25000$$

$$\begin{aligned}\text{Area of triangle } F &= \left[\frac{3}{2}\left(\sqrt{2}-1\right)\right]^2\\ &= \frac{9}{4}\left(3-2\sqrt{2}\right)\\ &= .38604\end{aligned}$$

We also have

$$\begin{aligned}\text{Area of polygon } A &= 1-2\times(\text{Area of triangle } B)\\ &= \sqrt{2}-\frac{1}{2}\\ &= .91421\end{aligned}$$

$$\begin{aligned}\text{Area of polygon } D &= 1-(\text{Area of triangle } C)\\ &= 1-\frac{1}{4}\\ &= \frac{3}{4}\\ &= .7500\end{aligned}$$

$$\begin{aligned}\text{Area of polygon } E &= 1-(\text{Area of triangle } F)\\ &= \frac{1}{4}\left(18\sqrt{2}-23\right)\\ &= .61396\end{aligned}$$

Referring back to Fig. 10.12.11, we see that

$a_{11,\,1}$ = Area of polygon A = .91421

$a_{10,\,1}$ = Area of triangle B = .04289

$a_{11,\,2}$ = Area of triangle C = .25000

$a_{10,\,2}$ = Area of polygon D = .75000

$a_{10,\,3}$ = Area of polygon E = .61396

By symmetry we then have

$$\begin{aligned}a_{11,\,1} &= a_{11,\,5} = a_{11,\,9} = a_{53} = a_{55} = a_{57}\\ &= .91421\end{aligned}$$

$$\begin{aligned}a_{10,\,1} &= a_{10,\,5} = a_{10,\,9} = a_{12,\,1} = a_{12,\,5} = a_{12,\,9}\\ &= a_{63} = a_{65} = a_{67} = a_{43} = a_{45}\\ &= a_{47} = .04289\end{aligned}$$

$$\begin{aligned}a_{11,\,2} &= a_{11,\,4} = a_{11,\,6} = a_{11,\,8}\\ &= a_{52} = a_{54} = a_{56} = a_{58} = .25000\end{aligned}$$

$$\begin{aligned}a_{10,\,2} &= a_{10,\,6} = a_{12,\,4} = a_{12,\,8}\\ &= a_{62} = a_{64} = a_{46} = a_{48} = .75000\end{aligned}$$

$$a_{10,\,3} = a_{12,\,7} = a_{61} = a_{49} = .61396.$$

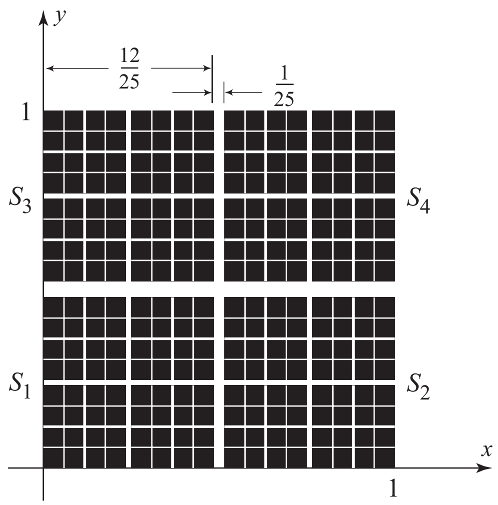
The remaining a_{ij} 's are all zero, and so the 12 beam equations (4) are

$$\begin{aligned}x_7 + x_8 + x_9 &= 13.00\\ x_4 + x_5 + x_6 &= 15.00\\ x_1 + x_2 + x_3 &= 8.00\\ 0.04289(x_3 + x_5 + x_7) + 0.75(x_6 + x_8) + 0.61396x_9 &= 14.79\\ 0.91421(x_3 + x_5 + x_7) + 0.25(x_2 + x_4 + x_6 + x_8) &= 14.31\\ 0.04289(x_3 + x_5 + x_7) + 0.75(x_2 + x_4) + 0.61396x_1 &= 3.81\\ x_3 + x_6 + x_9 &= 18.00\\ x_2 + x_5 + x_8 &= 12.00\\ x_1 + x_4 + x_7 &= 6.00\\ 0.04289(x_1 + x_5 + x_9) + 0.75(x_2 + x_6) + 0.61396x_3 &= 10.51\\ 0.91421(x_1 + x_5 + x_9) + 0.25(x_2 + x_4 + x_6 + x_8) &= 16.13\\ 0.04289(x_1 + x_5 + x_9) + 0.75(x_4 + x_8) + 0.61396x_7 &= 7.04\end{aligned}$$

Section 10.13

Exercise Set 10.13

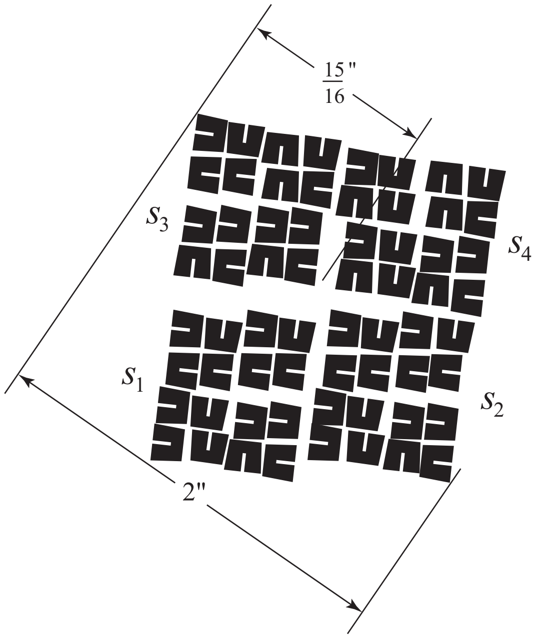
1. Each of the subsets S_1, S_2, S_3, S_4 in the figure is congruent to the entire set scaled by a factor of $\frac{12}{25}$. Also, the rotation angles for the four subsets are all 0°. The displacement distances can be determined from the figure to find the four similitudes that map the entire set onto the four subsets S_1, S_2, S_3, S_4. These are, respectively,



$T_i\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{12}{25}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e_i \\ f_i \end{bmatrix}$, $i = 1, 2, 3, 4$, where the four values of $\begin{bmatrix} e_i \\ f_i \end{bmatrix}$ are $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} \frac{13}{25} \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \frac{13}{25} \end{bmatrix}$, and $\begin{bmatrix} \frac{13}{25} \\ \frac{13}{25} \end{bmatrix}$.

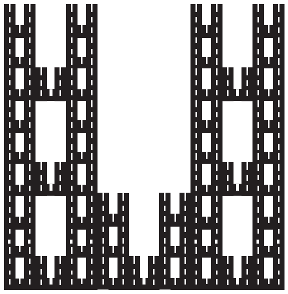
Because $s = \frac{12}{25}$ and $k = 4$ in the definition of a self-similar set, the Hausdorff dimension of the set is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(4)}{\ln\left(\frac{25}{12}\right)} = 1.888....$ The set is a fractal because its Hausdorff dimension is not an integer.

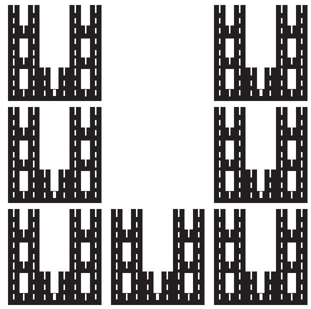
2. The rough measurements indicated in the figure give an approximate scale factor of $s \approx \frac{\left(\frac{15}{16}\right)}{2} = .47$ to two decimal places. Since $k = 4$, the Hausdorff dimension of the set is approximately $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(4)}{\ln\left(\frac{1}{.47}\right)} = 1.8$ to two significant digits. Examination of the figure reveals rotation angles of 180°, 180°, 0°, and –90° for the sets S_1, S_2, S_3, and S_4, respectively.



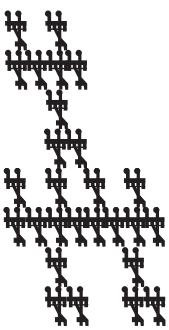
3. By inspection, reading left to right and top to bottom, the triplets are:
(0, 0, 0) none are rotated
(1, 0, 0) the upper right iteration is rotated 90°
(2, 0, 0) the upper right iteration is rotated 180°
(3, 0, 0) the upper right iteration is rotated 270°
(0, 0, 1) the lower right iteration is rotated 90°
(0, 0, 2) the lower right iteration is rotated 180°
(1, 2, 0) the upper right iteration is rotated 90° and the lower left is rotated 180°
(2, 1, 3) the upper right iteration is rotated 180°, the lower left is rotated 90°, and the lower right is rotated 270°
(2, 0, 1) the upper right iteration is rotated 180° and the lower right is rotated 90°
(2, 0, 2) the upper right and lower right iterations are both rotated 180°
(2, 2, 0) the upper right and lower left iterations are both rotated 180°
(0, 3, 3) the lower left and lower right iterations are both rotated 270°

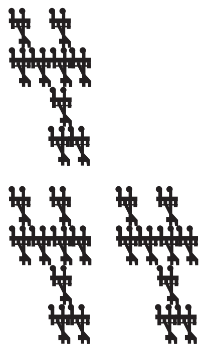
4. (a) The figure shows the original self-similar set and a decomposition of the set into seven nonoverlapping congruent subsets, each of which is congruent to the original set scaled by a factor $s = \frac{1}{3}$. By inspection, the rotations angles are 0° for all seven subsets. The Hausdorff dimension of the set is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(7)}{\ln(3)} = 1.771\ldots.$ Because its Hausdorff dimension is not an integer, the set is a fractal.



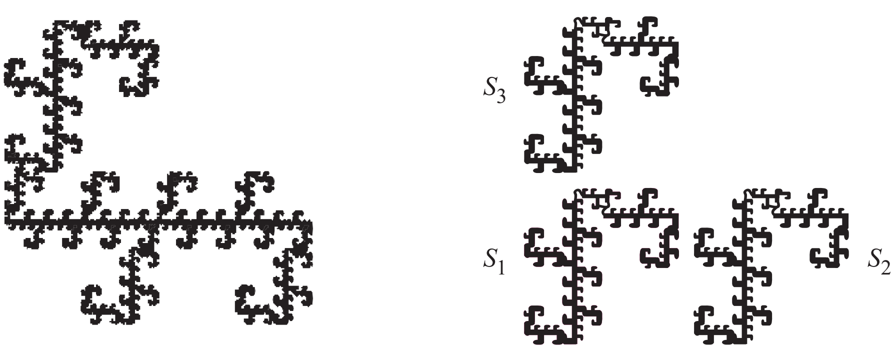


(b) The figure shows the original self-similar set and a decomposition of the set into three nonoverlapping congruent subsets, each of which is congruent to the original set scaled by a factor $s = \frac{1}{2}$. By inspection, the rotation angles are 180° for all three subsets. The Hausdorff dimension of the set is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(3)}{\ln(2)} = 1.584....$ Because its Hausdorff dimension is not an integer, the set is a fractal.

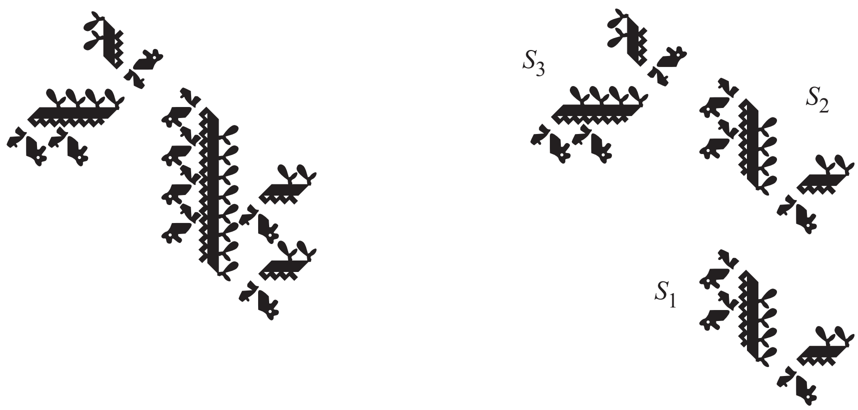




(c) The figure shows the original self-similar set and a decomposition of the set into three nonoverlapping congruent subsets, each of which is congruent to the original set scaled by a factor $s = \frac{1}{2}$. By inspection, the rotation angles are 180°, 180°, and −90° for S_1, S_2, and S_3, respectively. The Hausdorff dimension of the set is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(3)}{\ln(2)} = 1.584....$ Because its Hausdorff dimension is not an integer, the set is a fractal.



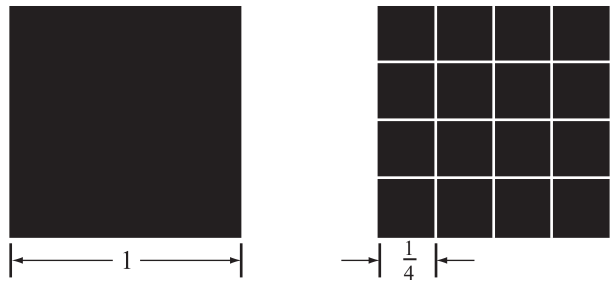
(d) The figure shows the original self-similar set and a decomposition of the set into three nonoverlapping congruent subsets, each of which is congruent to the original set scaled by a factor $s = \frac{1}{2}$. By inspection, the rotation angles are 180°, 180°, and −90° for S_1, S_2, and S_3, respectively. The Hausdorff dimension of the set is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(3)}{\ln(2)} = 1.584....$ Because its Hausdorff dimension is not an integer, the set is a fractal.



5. The matrix of the affine transformation in question is $\begin{bmatrix} .85 & .04 \\ -.04 & .85 \end{bmatrix}$. The matrix portion of a similitude is of the form $s\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Consequently, we must have $s \cos\theta = .85$ and $s \sin\theta = -.04$. Solving this pair of equations gives $s = \sqrt{(.85)^2 + (-.04)^2} = .8509...$ and $\theta = \tan^{-1}\left(\frac{-.04}{.85}\right) = -2.69...°$.

6. Letting $\begin{bmatrix} x \\ y \end{bmatrix}$ be the vector to the tip of the fern and using the hint, we have $\begin{bmatrix} x \\ y \end{bmatrix} = T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ or $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .85 & .04 \\ -.04 & .85 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} .075 \\ .180 \end{bmatrix}$. Solving this matrix equation gives $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .15 & -.04 \\ .04 & .15 \end{bmatrix}^{-1}\begin{bmatrix} .075 \\ .180 \end{bmatrix} = \begin{bmatrix} .766 \\ .996 \end{bmatrix}$ rounded to three decimal places.

7. As the figure indicates, the unit square can be expressed as the union of 16 nonoverlapping congruent squares, each of side length $\frac{1}{4}$. Consequently, the Hausdorff dimension of the unit square as given by Equation (2) of the text is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(16)}{\ln(4)} = 2.$

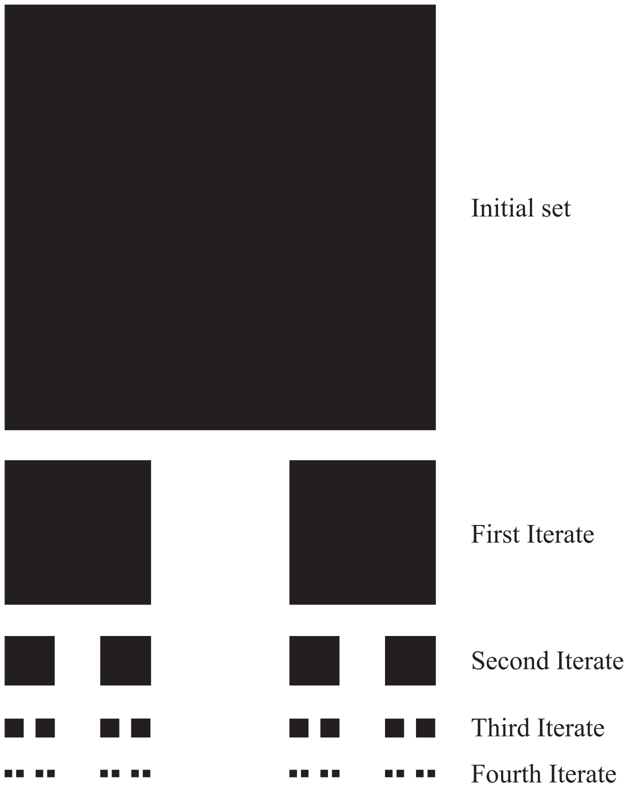


8. The similitude T_1 maps the unit square (whose vertices are (0, 0), (1, 0), (1, 1), and (0, 1)) onto the square whose vertices are (0, 0), $\left(\frac{3}{4}, 0\right)$, $\left(\frac{3}{4}, \frac{3}{4}\right)$, and $\left(0, \frac{3}{4}\right)$. The similitude T_2 maps the unit square onto the square whose vertices are $\left(\frac{1}{4}, 0\right)$, (1, 0), $\left(1, \frac{3}{4}\right)$, and $\left(\frac{1}{4}, \frac{3}{4}\right)$. The similitude T_3 maps the unit square onto the square whose vertices are $\left(0, \frac{1}{4}\right)$, $\left(\frac{3}{4}, \frac{1}{4}\right)$, $\left(\frac{3}{4}, 1\right)$, and (0, 1). Finally the similitude T_4 maps the unit square onto the square whose vertices are $\left(\frac{1}{4}, \frac{1}{4}\right)$, $\left(1, \frac{1}{4}\right)$, (1, 1), and $\left(\frac{1}{4}, 1\right)$. Each of these four smaller squares has side length of $\frac{3}{4}$, so that the common scale factor of the similitudes is $s = \frac{3}{4}$. The right-hand side of Equation (2) of the text gives $\frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(4)}{\ln\left(\frac{4}{3}\right)} = 4.818....$ This is not the correct Hausdorff dimension of the square (which is 2) because the four smaller squares overlap.

9. Because $s = \frac{1}{2}$ and $k = 8$, Equation (2) of the text gives $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(8)}{\ln(2)} = 3$ for the Hausdorff dimension of a unit cube. Because the Hausdorff dimension of the cube is the same as its topological dimension, the cube is not a fractal.

10. A careful examination of Figure Ex-10 in the text shows that the Menger sponge can be expressed as the union of 20 smaller nonoverlapping congruent Menger sponges each of side length $\frac{1}{3}$. Consequently, $k = 20$ and $s = \frac{1}{3}$, and so the Hausdorff dimension of the Menger sponge is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(20)}{\ln(3)} = 2.726....$ Because its Hausdorff dimension is not an integer, the Menger sponge is a fractal.

11. The figure shows the first four iterates as determined by Algorithm 1 and starting with the unit square as the initial set. Because $k = 2$ and $s = \frac{1}{3}$, the Hausdorff dimension of the Cantor set is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(2)}{\ln(3)} = 0.6309....$ Notice that the Cantor set is a subset of the unit interval along the x-axis and that its topological dimension must be 0 (since the topological dimension of any set is a nonnegative integer less than or equal to its Hausdorff dimension).



12. The area of the unit square S_0 is, of course, 1. Each of the eight similitudes $T_1, T_2, \ldots, T_8$ given in Equation (8) of the text has scale factor $s = \frac{1}{3}$, and so each maps the unit square onto a smaller square of area $\frac{1}{9}$. Because these eight smaller squares are nonoverlapping, their total area is $\frac{8}{9}$, which is then the area of the set S_1. By a similar argument, the area of the set S_2 is $\frac{8}{9}$-th the area of the set S_1. Continuing the argument further, we find that the areas of $S_0, S_1, S_2, S_3, S_4, \ldots,$ form the geometric sequence $1, \frac{8}{9}, \left(\frac{8}{9}\right)^2, \left(\frac{8}{9}\right)^3, \left(\frac{8}{9}\right)^4, \ldots.$ (Notice that this implies that the area of the Sierpinski carpet is 0, since the limit of $\left(\frac{8}{9}\right)^n$ as n tends to infinity is 0.)

Section 10.14

Exercise Set 10.14

1. Because $250 = 2 \cdot 5^3$ it follows from (i) that $\Pi(250) = 3 \cdot 250 = 750$.
Because $25 = 5^2$ it follows from (ii) that $\Pi(25) = 2 \cdot 25 = 50$.
Because $125 = 5^3$ it follows from (ii) that $\Pi(125) = 2 \cdot 125 = 250$.
Because $30 = 6 \cdot 5$ it follows from (ii) that $\Pi(30) = 2 \cdot 30 = 60$.
Because $10 = 2 \cdot 5$ it follows from (i) that $\Pi(10) = 3 \cdot 10 = 30$.

Because $50 = 2 \cdot 5^2$ it follows from (i) that $\Pi(50) = 3 \cdot 50 = 150$.

Because $3750 = 6 \cdot 5^4$ it follows from (ii) that $\Pi(3750) = 2 \cdot 3750 = 7500$.

Because $6 = 6 \cdot 5^0$ it follows from (ii) that $\Pi(6) = 2 \cdot 6 = 12$

Because $5 = 5^1$ it follows from (ii) that $\Pi(5) = 2 \cdot 5 = 10$.

2. The point (0, 0) is obviously a 1-cycle. We now choose another of the 36 points of the form $\left(\frac{m}{6}, \frac{n}{6}\right)$, say $\left(0, \frac{1}{6}\right)$. Its iterates produce the 12-cycle

$$\begin{bmatrix} 0 \\ \frac{1}{6} \end{bmatrix} \to \begin{bmatrix} \frac{1}{6} \\ \frac{2}{6} \end{bmatrix} \to \begin{bmatrix} \frac{3}{6} \\ \frac{5}{6} \end{bmatrix} \to \begin{bmatrix} \frac{2}{6} \\ \frac{1}{6} \end{bmatrix} \to \begin{bmatrix} \frac{3}{6} \\ \frac{4}{6} \end{bmatrix} \to \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix}$$

$$\to \begin{bmatrix} 0 \\ \frac{5}{6} \end{bmatrix} \to \begin{bmatrix} \frac{5}{6} \\ \frac{4}{6} \end{bmatrix} \to \begin{bmatrix} \frac{3}{6} \\ \frac{1}{6} \end{bmatrix} \to \begin{bmatrix} \frac{4}{6} \\ \frac{5}{6} \end{bmatrix} \to \begin{bmatrix} \frac{3}{6} \\ \frac{2}{6} \end{bmatrix} \to \begin{bmatrix} \frac{5}{6} \\ \frac{1}{6} \end{bmatrix}.$$

So far we have accounted for 13 of the 36 points. Taking one of the remaining points, say $\left(0, \frac{2}{6}\right)$, we arrive at the 4-cycle $\begin{bmatrix} 0 \\ \frac{2}{6} \end{bmatrix} \to \begin{bmatrix} \frac{2}{6} \\ \frac{4}{6} \end{bmatrix} \to \begin{bmatrix} 0 \\ \frac{4}{6} \end{bmatrix} \to \begin{bmatrix} \frac{4}{6} \\ \frac{2}{6} \end{bmatrix}$. We continue in this way, each time starting with some point of the form $\left(\frac{m}{6}, \frac{n}{6}\right)$ that has not yet appeared in a cycle, until we exhaust all such points. This yields a 3-cycle:

$\begin{bmatrix} \frac{3}{6} \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{3}{6} \\ \frac{3}{6} \end{bmatrix} \to \begin{bmatrix} 0 \\ \frac{3}{6} \end{bmatrix}$; another 4-cycle: $\begin{bmatrix} \frac{4}{6} \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{4}{6} \\ \frac{4}{6} \end{bmatrix} \to \begin{bmatrix} \frac{2}{6} \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{2}{6} \\ \frac{2}{6} \end{bmatrix}$; and another 12-cycle:

$$\begin{bmatrix} \frac{1}{6} \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} \to \begin{bmatrix} \frac{2}{6} \\ \frac{3}{6} \end{bmatrix} \to \begin{bmatrix} \frac{5}{6} \\ \frac{2}{6} \end{bmatrix} \to \begin{bmatrix} \frac{1}{6} \\ \frac{3}{6} \end{bmatrix} \to \begin{bmatrix} \frac{4}{6} \\ \frac{1}{6} \end{bmatrix}$$

$$\to \begin{bmatrix} \frac{5}{6} \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{5}{6} \\ \frac{5}{6} \end{bmatrix} \to \begin{bmatrix} \frac{4}{6} \\ \frac{3}{6} \end{bmatrix} \to \begin{bmatrix} \frac{1}{6} \\ \frac{4}{6} \end{bmatrix} \to \begin{bmatrix} \frac{5}{6} \\ \frac{3}{6} \end{bmatrix}$$

$$\to \begin{bmatrix} \frac{2}{6} \\ \frac{5}{6} \end{bmatrix}.$$

The possible periods of points for the form $\left(\frac{m}{6}, \frac{n}{6}\right)$ are thus 1, 3, 4 and 12. The least common multiple of these four numbers is 12, and so $\Pi(6) = 12$.

3. (a) We are given that $x_0 = 3$ and $x_1 = 7$. With $p = 15$ we have

$$\begin{aligned}
x_2 &= x_1 + x_0 \mod 15 = 7+3 \mod 15 = 10 \mod 15 = 10,\\
x_3 &= x_2 + x_1 \mod 15 = 10+7 \mod 15 = 17 \mod 15 = 2,\\
x_4 &= x_3 + x_2 \mod 15 = 2+10 \mod 15 = 12 \mod 15 = 12\\
x_5 &= x_4 + x_3 \mod 15 = 12+2 \mod 15 = 14 \mod 15 = 14,\\
x_6 &= x_5 + x_4 \mod 15 = 14+12 \mod 15 = 26 \mod 15 = 11,\\
x_7 &= x_6 + x_5 \mod 15 = 11+14 \mod 15 = 25 \mod 15 = 10\\
x_8 &= x_7 + x_6 \mod 15 = 10+11 \mod 15 = 21 \mod 15 = 6,\\
x_9 &= x_8 + x_7 \mod 15 = 6+10 \mod 15 = 16 \mod 15 = 1,\\
x_{10} &= x_9 + x_8 \mod 15 = 1+6 \mod 15 = 7 \mod 15 = 7,\\
x_{11} &= x_{10} + x_9 \mod 15 = 7+1 \mod 15 = 8 \mod 15 = 8\\
x_{12} &= x_{11} + x_{10} \mod 15 = 8+7 \mod 15 = 15 \mod 15 = 0,\\
x_{13} &= x_{12} + x_{11} \mod 15 = 0+8 \mod 15 = 8 \mod 15 = 8,\\
x_{14} &= x_{13} + x_{12} \mod 15 = 8+0 \mod 15 = 8 \mod 15 = 8,\\
x_{15} &= x_{14} + x_{13} \mod 15 = 8+8 \mod 15 = 16 \mod 15 = 1,\\
x_{16} &= x_{15} + x_{14} \mod 15 = 1+8 \mod 15 = 9 \mod 15 = 9,\\
x_{17} &= x_{16} + x_{15} \mod 15 = 9+1 \mod 15 = 10 \mod 15 = 10,\\
x_{18} &= x_{17} + x_{16} \mod 15 = 10+9 \mod 15 = 19 \mod 15 = 4,\\
x_{19} &= x_{18} + x_{17} \mod 15 = 4+10 \mod 15 = 14 \mod 15 = 14,\\
x_{20} &= x_{19} + x_{18} \mod 15 = 14+4 \mod 15 = 18 \mod 15 = 3,\\
x_{21} &= x_{20} + x_{19} \mod 15 = 3+14 \mod 15 = 17 \mod 15 = 2,\\
x_{22} &= x_{21} + x_{20} \mod 15 = 2+3 \mod 15 = 5 \mod 15 = 5,\\
x_{23} &= x_{22} + x_{21} \mod 15 = 5+2 \mod 15 = 7 \mod 15 = 7,\\
x_{24} &= x_{23} + x_{22} \mod 15 = 7+5 \mod 15 = 12 \mod 15 = 12,\\
x_{25} &= x_{24} + x_{23} \mod 15 = 12+7 \mod 15 = 19 \mod 15 = 4,\\
x_{26} &= x_{25} + x_{24} \mod 15 = 4+12 \mod 15 = 16 \mod 15 = 1,\\
x_{27} &= x_{26} + x_{25} \mod 15 = 1+4 \mod 15 = 5 \mod 15 = 5,\\
x_{28} &= x_{27} + x_{26} \mod 15 = 5+1 \mod 15 = 6 \mod 15 = 6,\\
x_{29} &= x_{28} + x_{27} \mod 15 = 6+5 \mod 15 = 11 \mod 15 = 11,\\
x_{30} &= x_{29} + x_{28} \mod 15 = 11+6 \mod 15 = 17 \mod 15 = 2,\\
x_{31} &= x_{30} + x_{29} \mod 15 = 2+11 \mod 15 = 13 \mod 15 = 13,\\
x_{32} &= x_{31} + x_{30} \mod 15 = 13+2 \mod 15 = 15 \mod 15 = 0,\\
x_{33} &= x_{32} + x_{31} \mod 15 = 0+13 \mod 15 = 13 \mod 15 = 13,\\
x_{34} &= x_{33} + x_{32} \mod 15 = 13+0 \mod 15 = 13 \mod 15 = 13,\\
x_{35} &= x_{34} + x_{33} \mod 15 = 13+13 \mod 15 = 26 \mod 15 = 11,\\
x_{36} &= x_{35} + x_{34} \mod 15 = 11+13 \mod 15 = 24 \mod 15 = 9,\\
x_{37} &= x_{36} + x_{35} \mod 15 = 9+11 \mod 15 = 20 \mod 15 = 5,\\
x_{38} &= x_{37} + x_{36} \mod 15 = 5+9 \mod 15 = 14 \mod 15 = 14,\\
x_{39} &= x_{38} + x_{37} \mod 15 = 14+5 \mod 15 = 19 \mod 15 = 4,\\
x_{40} &= x_{39} + x_{38} \mod 15 = 4+14 \mod 15 = 18 \mod 15 = 3,\\
x_{41} &= x_{40} + x_{39} \mod 15 = 3+4 \mod 15 = 7 \mod 15 = 7,
\end{aligned}$$

and finally: $x_{40} = x_0$ and $x_{41} = x_1$. Thus this sequence is periodic with period 40.

(b) Step (ii) of the algorithm is
$x_{n+1} = x_n + x_{n-1} \bmod p$.
Replacing n in this formula by $n + 1$ gives

$$\begin{aligned} x_{n+2} &= x_{n+1} + x_n \bmod p \\ &= (x_n + x_{n-1}) + x_n \bmod p \\ &= 2x_n + x_{n-1} \bmod p. \end{aligned}$$

These equations can be written as

$$x_{n+1} = x_{n-1} + x_n \bmod p$$
$$x_{n+2} = x_{n-1} + 2x_n \bmod p$$

which in matrix form are

$$\begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} \bmod p.$$

(c) Beginning with $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$, we obtain

$$\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} \bmod 21 = \begin{bmatrix} 10 \\ 15 \end{bmatrix} \bmod 21 = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 15 \end{bmatrix} \bmod 21 = \begin{bmatrix} 25 \\ 40 \end{bmatrix} \bmod 21 = \begin{bmatrix} 4 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 19 \end{bmatrix} \bmod 21 = \begin{bmatrix} 23 \\ 42 \end{bmatrix} \bmod 21 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \bmod 21 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \bmod 21 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x_{10} \\ x_{11} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \bmod 21 = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \bmod 21 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} \bmod 21 = \begin{bmatrix} 10 \\ 16 \end{bmatrix} \bmod 21 = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} x_{14} \\ x_{15} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 16 \end{bmatrix} \bmod 21 = \begin{bmatrix} 26 \\ 42 \end{bmatrix} \bmod 21 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_{16} \\ x_{17} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} \bmod 21 = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \bmod 21 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

and we see that $\begin{bmatrix} x_{16} \\ x_{17} \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$.

4. (c) We have that

$$C\left(\begin{bmatrix} \frac{1}{101} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{101} \\ 0 \end{bmatrix} \bmod 1 = \begin{bmatrix} \frac{1}{101} \\ \frac{1}{101} \end{bmatrix},$$

$$C^2\left(\begin{bmatrix} \frac{1}{101} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{101} \\ \frac{1}{101} \end{bmatrix} \bmod 1 = \begin{bmatrix} \frac{2}{101} \\ \frac{3}{101} \end{bmatrix},$$

$$C^3\left(\begin{bmatrix} \frac{1}{101} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{101} \\ \frac{3}{101} \end{bmatrix} \bmod 1 = \begin{bmatrix} \frac{5}{101} \\ \frac{8}{101} \end{bmatrix},$$

$$C^4\left(\begin{bmatrix} \frac{1}{101} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{5}{101} \\ \frac{8}{101} \end{bmatrix} \bmod 1 = \begin{bmatrix} \frac{13}{101} \\ \frac{21}{101} \end{bmatrix},$$

$$C^5\left(\begin{bmatrix} \frac{1}{101} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{13}{101} \\ \frac{21}{101} \end{bmatrix} \bmod 1 = \begin{bmatrix} \frac{34}{101} \\ \frac{55}{101} \end{bmatrix}.$$

Because all five iterates are different, the period of the periodic point $\left(\frac{1}{101}, 0\right)$ must be greater than 5.

5. If $0 \le x < 1$ and $0 \le y < 1$, then

$T(x, y) = \left(x + \frac{5}{12}, y\right) \bmod 1$, and so

$T^2(x, y) = \left(x + \frac{10}{12}, y\right) \bmod 1$,

$T^3(x, y) = \left(x + \frac{15}{12}, y\right) \bmod 1, \ldots,$

$T^{12}(x, y) = \left(x + \frac{60}{12}, y\right) \bmod 1$
$= (x+5, y) \bmod 1 = (x, y).$

Thus every point in S returns to its original position after 12 iterations and so every point in S is a periodic point with period at most 12. Because every point is a periodic point, no point can have a dense set of iterates, and so the mapping cannot be chaotic.

6. (a) The matrix of Arnold's cat map, $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, is one in which (i) the entries are all integers, (ii) the determinant is 1, and (iii) the eigenvalues, $\frac{3+\sqrt{5}}{2} = 2.6180\ldots$ and $\frac{3-\sqrt{5}}{2} = 0.3819\ldots,$ do not have magnitude 1. The three conditions of an Anosov automorphism are thus satisfied.

(b) The eigenvalues of the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are ± 1, both of which have magnitude 1. By part (iii) of the definition of an Anosov automorphism, this matrix is not the matrix of an Anosov automorphism.

The entries of the matrix $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ are integers, its determinant is 1; and neither of its eigenvalues, $2 \pm \sqrt{3}$, has magnitude 1. Consequently, this is the matrix of an Anosov automorphism.

The eigenvalues of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are both equal to 1, and so both have magnitude 1. By part (iii) of the definition, this is not the matrix of an Anosov automorphism.

The entries of the matrix $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ are integers; its determinant is 1; and neither of its eigenvalues, $4 \pm \sqrt{15}$, has magnitude 1. Consequently, this is the matrix of an Anosov automorphism.

The determinant of the matrix $\begin{bmatrix} 6 & 2 \\ 5 & 2 \end{bmatrix}$ is 2, and so by part (ii) of the definition, this is not the matrix of an Anosov automorphism.

(c) The eigenvalues of the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are $\pm i$; both of which have magnitude 1. By part (iii) of the definition, this cannot be the matrix of an Anosov automorphism.

Starting with an arbitrary point $\begin{bmatrix} x \\ y \end{bmatrix}$ in the interior of S, (that is, with $0 < x < 1$ and $0 < y < 1$) we obtain

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \bmod 1 = \begin{bmatrix} y \\ -x \end{bmatrix} \bmod 1 = \begin{bmatrix} y \\ 1-x \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ 1-x \end{bmatrix} \bmod 1 = \begin{bmatrix} 1-x \\ -y \end{bmatrix} \bmod 1 = \begin{bmatrix} 1-x \\ 1-y \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1-x \\ 1-y \end{bmatrix} \bmod 1 = \begin{bmatrix} 1-y \\ -1+x \end{bmatrix} \bmod 1 = \begin{bmatrix} 1-y \\ x \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1-y \\ x \end{bmatrix} \bmod 1 = \begin{bmatrix} x \\ -1+y \end{bmatrix} \bmod 1 = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus every point in the interior of S is a periodic point with period at most 4. The geometric effect of this transformation, as seen by the iterates, is to rotate each point in the interior of S clockwise by 90° about the center point $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ of S. Consequently, each point in the interior of S has period 4 with the exception of the center point $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ which is a fixed point.

For points not in the interior of S, we first observe that the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a fixed point, which can easily be verified. Starting with a point of the form $\begin{bmatrix} x \\ 0 \end{bmatrix}$ with $0 < x < 1$, we obtain a 4-cycle

$\begin{bmatrix} x \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1-x \end{bmatrix} \to \begin{bmatrix} 1-x \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ x \end{bmatrix} \to \begin{bmatrix} x \\ 0 \end{bmatrix}$ if

$x \neq \frac{1}{2}$, otherwise we obtain the 2-cycle

$\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$

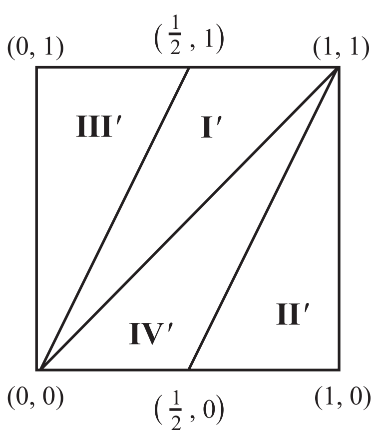
Similarly, starting with a point of the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$ with $0 < y < 1$, we obtain a 4-cycle

$\begin{bmatrix} 0 \\ y \end{bmatrix} \to \begin{bmatrix} y \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1-y \end{bmatrix} \to \begin{bmatrix} 1-y \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ y \end{bmatrix}$ if

$y \neq \frac{1}{2}$, otherwise we obtain the 2-cycle

$\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$. Thus every point not in the interior of S is a periodic point with 1, 2, or 4. Finally because no point in S can have a dense set of iterates, it follows that the mapping cannot be chaotic.

9. As per the hint, we wish to find the regions in S that map onto the four indicated regions in the figure below.



We first consider region $\mathbf{I}'$ with vertices (0, 0), $\left(\frac{1}{2}, 1\right)$, and (1, 1). We seek points (x_1, y_1), (x_2, y_2), and (x_3, y_3), with entries that lie in [0, 1], that map onto these three points under the mapping $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$ for certain integer values of a and b to be determined. This leads to the three equations

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix},$

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

The inverse of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. We multiply the above three matrix equations by this inverse and set $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$. Notice that c and d must be integers. This leads to

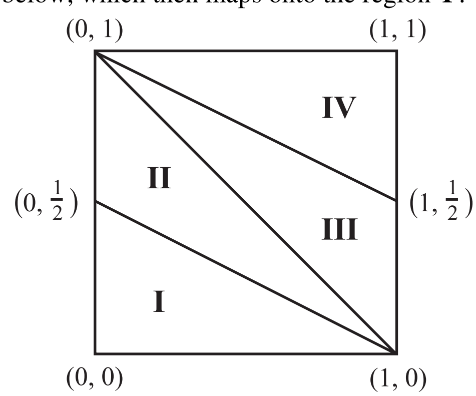
$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = -\begin{bmatrix} c \\ d \end{bmatrix},$

$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix},$

$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix}.$

The only integer values of c and d that will give values of x_i and y_i in the interval [0, 1] are $c = d = 0$. This then gives $a = b = 0$ and the mapping $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ maps the three points (0, 0), $\left(0, \frac{1}{2}\right)$, and (1, 0) to the three points (0, 0), $\left(\frac{1}{2}, 1\right)$, and (1, 1), respectively.

The three points (0, 0), $\left(0, \frac{1}{2}\right)$, and (1, 0) define the triangular region labeled **I** in the diagram below, which then maps onto the region $\mathbf{I}'$.



For region $\mathbf{II}'$, the calculations are as follows:

$$\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ y_1\end{bmatrix}+\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix},$$

$$\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x_2\\ y_2\end{bmatrix}+\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix},$$

$$\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x_3\\ y_3\end{bmatrix}+\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix};$$

$$\begin{bmatrix}x_1\\ y_1\end{bmatrix}=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix}=\begin{bmatrix}1\\ -\frac{1}{2}\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix},$$

$$\begin{bmatrix}x_2\\ y_2\end{bmatrix}=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix},$$

$$\begin{bmatrix}x_3\\ y_3\end{bmatrix}=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix}=\begin{bmatrix}2\\ -1\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix}.$$

Only $c = 1$ and $d = -1$ will work. This leads to $a = 0$, $b = -1$ and the mapping $\begin{bmatrix}x\\ y\end{bmatrix}\to\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\begin{bmatrix}0\\ -1\end{bmatrix}$ maps region **II** with vertices $\left(0, \frac{1}{2}\right)$, (0, 1), and (1, 0) onto region **II′**. For region **III′**, the calculations are as follows:

$$\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ y_1\end{bmatrix}+\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix},$$

$$\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x_2\\ y_2\end{bmatrix}+\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix},$$

$$\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x_3\\ y_3\end{bmatrix}+\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix};$$

$$\begin{bmatrix}x_1\\ y_1\end{bmatrix}=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}0\\ 0\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix}=-\begin{bmatrix}c\\ d\end{bmatrix},$$

$$\begin{bmatrix}x_2\\ y_2\end{bmatrix}=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix}=\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix},$$

$$\begin{bmatrix}x_3\\ y_3\end{bmatrix}=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix}=\begin{bmatrix}-1\\ 1\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix}.$$

Only $c = -1$ and $d = 0$ will work. This leads to $a = -1$, $b = -1$ and the mapping $\begin{bmatrix}x\\ y\end{bmatrix}\to\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\begin{bmatrix}-1\\ -1\end{bmatrix}$ maps region **III** with vertices (1, 0), $\left(1, \frac{1}{2}\right)$, and (0, 1) onto region **III′**.

For region **IV′**, the calculations are as follows:

$$\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ y_1\end{bmatrix}+\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix},$$

$$\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x_2\\ y_2\end{bmatrix}+\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix},$$

$$\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x_3\\ y_3\end{bmatrix}+\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix};$$

$$\begin{bmatrix}x_1\\ y_1\end{bmatrix}=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}0\\ 0\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix}=-\begin{bmatrix}c\\ d\end{bmatrix},$$

$$\begin{bmatrix}x_2\\ y_2\end{bmatrix}=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix},$$

$$\begin{bmatrix}x_3\\ y_3\end{bmatrix}=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix}=\begin{bmatrix}1\\ -\frac{1}{2}\end{bmatrix}-\begin{bmatrix}c\\ d\end{bmatrix}.$$

Only $c = 0$ and $d = -1$ will work. This leads to $a = -1$, $b = -2$ and the mapping $\begin{bmatrix}x\\ y\end{bmatrix}\to\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\begin{bmatrix}-1\\ -2\end{bmatrix}$ maps region **IV** with vertices (0, 1), (1, 1), and $\left(1, \frac{1}{2}\right)$ onto region **IV′**.

12. As per the hint, we want to find all solutions of the matrix equation $\begin{bmatrix}x_0\\ y_0\end{bmatrix}=\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}x_0\\ y_0\end{bmatrix}-\begin{bmatrix}r\\ s\end{bmatrix}$ where $0 \le x_0 < 1$, $0 \le y_0 < 1$, and r and s are nonnegative integers. This equation can be rewritten as $\begin{bmatrix}1 & 3\\ 3 & 4\end{bmatrix}\begin{bmatrix}x_0\\ y_0\end{bmatrix}=\begin{bmatrix}r\\ s\end{bmatrix}$, which has the solution $x_0=\frac{-4r+3s}{5}$ and $y_0=\frac{3r-s}{5}$. First trying $r = 0$ and $s = 0, 1, 2, \ldots$, then $r = 1$ and $s = 0, 1, 2, \ldots$, etc., we find that the only values of r and s that yield values of x_0 and y_0 lying in [0, 1) are:

$r = 1$ and $s = 2$, which give $x_0=\frac{2}{5}$ and $y_0=\frac{1}{5}$;

$r = 2$ and $s = 3$, which give $x_0=\frac{1}{5}$ and $y_0=\frac{3}{5}$;

$r = 2$ and $s = 4$, which give $x_0=\frac{4}{5}$ and $y_0=\frac{2}{5}$;

$r = 3$ and $s = 5$, which give $x_0=\frac{3}{5}$ and $y_0=\frac{4}{5}$.

We can then check that $\left(\frac{2}{5}, \frac{1}{5}\right)$ and $\left(\frac{3}{5}, \frac{4}{5}\right)$ form one 2-cycle and $\left(\frac{1}{5}, \frac{3}{5}\right)$ and $\left(\frac{4}{5}, \frac{2}{5}\right)$ form another 2-cycle.

14. Begin with a 101×101 array of white pixels and add the letter 'A' in black pixels to it. Apply the mapping T to this image, which will scatter the black pixels throughout the image. Then superimpose the letter 'B' in black pixels onto this image. Apply the mapping T again and then superimpose the letter 'C' in black pixels onto the resulting image. Repeat this procedure with the letter 'D' and 'E'. The next application of the mapping will return you to the letter 'A' with the pixels for the letters 'B' through 'E' scattered in the background.

Section 10.15

Exercise Set 10.15

1. First group the plaintext into pairs, add the dummy letter T, and get the numerial equivalents from Table 1.

DA	*RK*	*NI*	*GH*	*TT*
4 1	18 11	14 9	7 8	20 20

(a) For the enciphering matrix $A=\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}$, reducing everything mod 26, we have

$$\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix}4\\ 1\end{bmatrix}=\begin{bmatrix}7\\ 9\end{bmatrix} \qquad \begin{matrix}G\\ I\end{matrix}$$

$$\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix}18\\ 11\end{bmatrix}=\begin{bmatrix}51\\ 47\end{bmatrix}=\begin{bmatrix}25\\ 21\end{bmatrix} \qquad \begin{matrix}Y\\ U\end{matrix}$$

$$\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix}14\\ 9\end{bmatrix}=\begin{bmatrix}41\\ 37\end{bmatrix}=\begin{bmatrix}15\\ 11\end{bmatrix} \qquad \begin{matrix}O\\ K\end{matrix}$$

$$\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix}7\\ 8\end{bmatrix}=\begin{bmatrix}31\\ 22\end{bmatrix}=\begin{bmatrix}5\\ 22\end{bmatrix} \qquad \begin{matrix}E\\ V\end{matrix}$$

$$\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix}20\\ 20\end{bmatrix}=\begin{bmatrix}80\\ 60\end{bmatrix}=\begin{bmatrix}2\\ 8\end{bmatrix} \qquad \begin{matrix}B\\ H\end{matrix}$$

The Hill cipher is *GIYUOKEVBH*.

(b) For the enciphering matrix $A=\begin{bmatrix}4 & 3\\ 1 & 2\end{bmatrix}$, reducing everything mod 26, we have

$$\begin{bmatrix}4 & 3\\ 1 & 2\end{bmatrix}\begin{bmatrix}4\\ 1\end{bmatrix}=\begin{bmatrix}19\\ 6\end{bmatrix} \qquad \begin{matrix}S\\ F\end{matrix}$$

$$\begin{bmatrix}4 & 3\\ 1 & 2\end{bmatrix}\begin{bmatrix}18\\ 11\end{bmatrix}=\begin{bmatrix}105\\ 40\end{bmatrix}=\begin{bmatrix}1\\ 14\end{bmatrix} \qquad \begin{matrix}A\\ N\end{matrix}$$

$$\begin{bmatrix}4 & 3\\ 1 & 2\end{bmatrix}\begin{bmatrix}14\\ 9\end{bmatrix}=\begin{bmatrix}83\\ 32\end{bmatrix}=\begin{bmatrix}5\\ 6\end{bmatrix} \qquad \begin{matrix}E\\ F\end{matrix}$$

$$\begin{bmatrix}4 & 3\\ 1 & 2\end{bmatrix}\begin{bmatrix}7\\ 8\end{bmatrix}=\begin{bmatrix}52\\ 23\end{bmatrix}=\begin{bmatrix}0\\ 23\end{bmatrix} \qquad \begin{matrix}Z\\ W\end{matrix}$$

$$\begin{bmatrix}4 & 3\\ 1 & 2\end{bmatrix}\begin{bmatrix}20\\ 20\end{bmatrix}=\begin{bmatrix}140\\ 60\end{bmatrix}=\begin{bmatrix}10\\ 8\end{bmatrix} \qquad \begin{matrix}J\\ H\end{matrix}$$

The Hill cipher is *SFANEFZWJH*.

2. (a) For $A=\begin{bmatrix}9 & 1\\ 7 & 2\end{bmatrix}$, $\det(A) = 18 - 7 = 11$, which is not divisible by 2 or 13. Therefore by Corollary 10.15.3, A is invertible. From Eqation (2):

$$\begin{aligned}A^{-1} &= (11)^{-1}\begin{bmatrix}2 & -1\\ -7 & 9\end{bmatrix}\\ &= 19\begin{bmatrix}2 & -1\\ -7 & 9\end{bmatrix}\\ &= \begin{bmatrix}38 & -19\\ -133 & 171\end{bmatrix}\\ &= \begin{bmatrix}12 & 7\\ 23 & 15\end{bmatrix} \pmod{26}.\end{aligned}$$

Checking:

$$\begin{aligned}AA^{-1} &= \begin{bmatrix}9 & 1\\ 7 & 2\end{bmatrix}\begin{bmatrix}12 & 7\\ 23 & 15\end{bmatrix}\\ &= \begin{bmatrix}131 & 78\\ 130 & 79\end{bmatrix}\\ &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \pmod{26}\end{aligned}$$

$$\begin{aligned}A^{-1}A &= \begin{bmatrix}12 & 7\\ 23 & 15\end{bmatrix}\begin{bmatrix}9 & 1\\ 7 & 2\end{bmatrix}\\ &= \begin{bmatrix}157 & 26\\ 312 & 53\end{bmatrix}\\ &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \pmod{26}.\end{aligned}$$

(b) For $A=\begin{bmatrix}3 & 1\\ 5 & 3\end{bmatrix}$, $\det(A) = 9 - 5 = 4$, which is divisible by 2. Therefore by Corollary 10.15.3, A is not invertible.

(c) For $A=\begin{bmatrix}8 & 11\\ 1 & 9\end{bmatrix}$,

$\det(A) = 72 - 11 = 61 = 9 \pmod{26}$, which is not divisible by 2 or 13. Therefore by Corollary 10.15.3, A is invertible. From (2):

$$\begin{aligned}A^{-1} &= (9)^{-1}\begin{bmatrix}9 & -11\\ -1 & 8\end{bmatrix}\\ &= 3\begin{bmatrix}9 & -11\\ -1 & 8\end{bmatrix}\\ &= \begin{bmatrix}27 & -33\\ -3 & 24\end{bmatrix}\\ &= \begin{bmatrix}1 & 19\\ 23 & 24\end{bmatrix} \pmod{26}.\end{aligned}$$

Checking:

$$AA^{-1} = \begin{bmatrix} 8 & 11 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} 1 & 19 \\ 23 & 24 \end{bmatrix} = \begin{bmatrix} 261 & 416 \\ 208 & 235 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{26}$$

$$A^{-1}A = \begin{bmatrix} 1 & 19 \\ 23 & 24 \end{bmatrix} \begin{bmatrix} 8 & 11 \\ 1 & 9 \end{bmatrix} = \begin{bmatrix} 27 & 182 \\ 208 & 469 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{26}.$$

(d) For $A = \begin{bmatrix} 2 & 1 \\ 1 & 7 \end{bmatrix}$, $\det(A) = 14 - 1 = 13$, which is divisible by 13. Therefore by Corollary 10.15.4, A is not invertible.

(e) For $A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$, $\det(A) = 6 - 6 = 0$, so that A is not invertible by Corollary 10.15.4.

(f) For $A = \begin{bmatrix} 1 & 8 \\ 1 & 3 \end{bmatrix}$, $\det(A) = 3 - 8 = -5 = 21$ (mod 26), which is not divisible by 2 or 13. Therefore by Corollary 10.15.4, A is invertible. From (2):

$$A^{-1} = (21)^{-1} \begin{bmatrix} 3 & -8 \\ -1 & 1 \end{bmatrix} = 5 \begin{bmatrix} 3 & -8 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & -40 \\ -5 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 21 & 5 \end{bmatrix} \pmod{26}.$$

Checking:

$$AA^{-1} = \begin{bmatrix} 1 & 8 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 15 & 12 \\ 21 & 5 \end{bmatrix} = \begin{bmatrix} 183 & 52 \\ 78 & 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{26}$$

$$A^{-1}A = \begin{bmatrix} 15 & 12 \\ 21 & 5 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 27 & 156 \\ 26 & 183 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{26}.$$

3. From Table 1 the numerical equivalent of this ciphertext is

19 1   11 14   15 24   1 15   10 24

Now we have to find the inverse of $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.

Since $\det(A) = 8 - 3 = 5$, we have by (2):

$$A^{-1} = (5)^{-1} \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} = 21 \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 42 & -21 \\ -63 & 84 \end{bmatrix} = \begin{bmatrix} 16 & 5 \\ 15 & 6 \end{bmatrix} \pmod{26}.$$

To obtain the plaintext, multiply each ciphertext vector by A^{-1} and reduce the results modulo 26.

$$\begin{bmatrix} 16 & 5 \\ 15 & 6 \end{bmatrix} \begin{bmatrix} 19 \\ 1 \end{bmatrix} = \begin{bmatrix} 309 \\ 291 \end{bmatrix} = \begin{bmatrix} 23 \\ 5 \end{bmatrix} \quad \begin{matrix} W \\ E \end{matrix}$$

$$\begin{bmatrix} 16 & 5 \\ 15 & 6 \end{bmatrix} \begin{bmatrix} 11 \\ 14 \end{bmatrix} = \begin{bmatrix} 246 \\ 249 \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \end{bmatrix} \quad \begin{matrix} L \\ O \end{matrix}$$

$$\begin{bmatrix} 16 & 5 \\ 15 & 6 \end{bmatrix} \begin{bmatrix} 15 \\ 24 \end{bmatrix} = \begin{bmatrix} 360 \\ 369 \end{bmatrix} = \begin{bmatrix} 22 \\ 5 \end{bmatrix} \quad \begin{matrix} V \\ E \end{matrix}$$

$$\begin{bmatrix} 16 & 5 \\ 15 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 15 \end{bmatrix} = \begin{bmatrix} 91 \\ 105 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \end{bmatrix} \quad \begin{matrix} M \\ A \end{matrix}$$

$$\begin{bmatrix} 16 & 5 \\ 15 & 6 \end{bmatrix} \begin{bmatrix} 10 \\ 24 \end{bmatrix} = \begin{bmatrix} 280 \\ 294 \end{bmatrix} = \begin{bmatrix} 20 \\ 8 \end{bmatrix} \quad \begin{matrix} T \\ H \end{matrix}$$

The plaintext is thus WE LOVE MATH.

4. From Table 1 the numerical equivalent of the known plaintext is

AR	*MY*
1 18	13 25

and the numerical equivalent of the corresponding ciphertext is

$\begin{matrix} SL & HK \\ 19\ \ 12 & 8\ \ 11 \end{matrix}$

so the corresponding plaintext and ciphertext vectors are

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 18 \end{bmatrix} \leftrightarrow \mathbf{c}_1 = \begin{bmatrix} 19 \\ 12 \end{bmatrix}$$

$$\mathbf{p}_2 = \begin{bmatrix} 13 \\ 25 \end{bmatrix} \leftrightarrow \mathbf{c}_2 = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

We want to reduce $C = \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \end{bmatrix} = \begin{bmatrix} 19 & 12 \\ 8 & 11 \end{bmatrix}$ to I by elementary row operations and simultaneously apply these operations to $P = \begin{bmatrix} \mathbf{p}_1^T \\ \mathbf{p}_2^T \end{bmatrix} = \begin{bmatrix} 1 & 18 \\ 13 & 25 \end{bmatrix}.$

The calculations are as follows:

$\left[\begin{array}{cc|cc} 19 & 12 & 1 & 18 \\ 8 & 11 & 13 & 25 \end{array}\right]$ Form the matrix $[C \mid P]$.

$\left[\begin{array}{cc|cc} 1 & 132 & 11 & 198 \\ 8 & 11 & 13 & 25 \end{array}\right]$ Multiply the first row by $19^{-1} = 11 \pmod{26}$.

$\left[\begin{array}{cc|cc} 1 & 2 & 11 & 16 \\ 8 & 11 & 13 & 25 \end{array}\right]$ Replace 132 and 198 by their residues modulo 26.

$\left[\begin{array}{cc|cc} 1 & 2 & 11 & 16 \\ 0 & -5 & -75 & -103 \end{array}\right]$ -8 times the first row to the second.

$\left[\begin{array}{cc|cc} 1 & 2 & 11 & 16 \\ 0 & 21 & 3 & 1 \end{array}\right]$ Replace the entries in the second row by their residues modulo 26.

$\left[\begin{array}{cc|cc} 1 & 2 & 11 & 16 \\ 0 & 1 & 15 & 5 \end{array}\right]$ Multiply the second row by $21^{-1} = 5 \pmod{26}$.

$\left[\begin{array}{cc|cc} 1 & 0 & -19 & 6 \\ 0 & 1 & 15 & 5 \end{array}\right]$ Add -2 times the second row to the first.

$\left[\begin{array}{cc|cc} 1 & 0 & 7 & 6 \\ 0 & 1 & 15 & 5 \end{array}\right]$ Replace -19 by its residue modulo 26.

Thus $(A^{-1})^T = \begin{bmatrix} 7 & 6 \\ 15 & 5 \end{bmatrix}$ so the deciphering matrix is $A^{-1} = \begin{bmatrix} 7 & 15 \\ 6 & 5 \end{bmatrix} \pmod{26}$.

Since $\det(A^{-1}) = 35 - 90 = -55 = 23 \pmod{26}$,

$$\begin{aligned} A &= (A^{-1})^{-1} \\ &= 23^{-1}\begin{bmatrix} 5 & -15 \\ -6 & 7 \end{bmatrix} \\ &= 17\begin{bmatrix} 5 & -15 \\ -6 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 85 & -255 \\ -102 & 119 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 5 \\ 2 & 15 \end{bmatrix} \pmod{26} \end{aligned}$$

is the enciphering matrix.

5. From Table 1 the numerical equivalent of the known plaintext is

$\begin{matrix} AT & OM \\ 1\ \ 20 & 15\ \ 13 \end{matrix}$

and the numerical equivalent of the corresponding ciphertext is

$\begin{matrix} JY & QO \\ 10\ \ 25 & 17\ \ 15 \end{matrix}$

The corresponding plaintext and ciphertext vectors are:

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 20 \end{bmatrix} \leftrightarrow \mathbf{c}_1 = \begin{bmatrix} 10 \\ 25 \end{bmatrix}$$

$$\mathbf{p}_2 = \begin{bmatrix} 15 \\ 13 \end{bmatrix} \leftrightarrow \mathbf{c}_2 = \begin{bmatrix} 17 \\ 15 \end{bmatrix}$$

We want to reduce $C = \begin{bmatrix} 10 & 25 \\ 17 & 15 \end{bmatrix}$ to I by elementary row operations and simultaneously apply these operations to $P = \begin{bmatrix} 1 & 20 \\ 15 & 13 \end{bmatrix}.$

The calculations are as follows:

$\left[\begin{array}{cc|cc} 10 & 25 & 1 & 20 \\ 17 & 15 & 15 & 13 \end{array}\right]$ Form the matrix $[C \mid P]$.

$\left[\begin{array}{cc|cc} 27 & 40 & 16 & 33 \\ 17 & 15 & 15 & 13 \end{array}\right]$ Add the second row to the first (since 10^{-1} does not exist mod 26).

$\left[\begin{array}{cc|cc} 1 & 14 & 16 & 7 \\ 17 & 15 & 15 & 13 \end{array}\right]$ Replace the entries in the first row by their residues modulo 26.

$\left[\begin{array}{cc|cc} 1 & 14 & 16 & 7 \\ 0 & -223 & -257 & -106 \end{array}\right]$ Add -17 times the first row to the second.

$\left[\begin{array}{cc|cc} 1 & 14 & 16 & 7 \\ 0 & 11 & 3 & 24 \end{array}\right]$ Replace the entries in the second row by their residues modulo 26.

$\left[\begin{array}{cc|cc} 1 & 14 & 16 & 7 \\ 0 & 1 & 57 & 456 \end{array}\right]$ Multiply the second row by $11^{-1} = 19 \pmod{26}$.

$\left[\begin{array}{cc|cc} 1 & 14 & 16 & 7 \\ 0 & 1 & 5 & 14 \end{array}\right]$ Replace the entries in the second row by their residues modulo 26.

$\left[\begin{array}{cc|cc} 1 & 0 & -54 & -189 \\ 0 & 1 & 5 & 14 \end{array}\right]$ Add −14 times the second row to the first.

$\left[\begin{array}{cc|cc} 1 & 0 & 24 & 19 \\ 0 & 1 & 5 & 14 \end{array}\right]$ Replace −54 and −189 by their residues modulo 26.

Thus $(A^{-1})^T = \begin{bmatrix} 24 & 19 \\ 5 & 14 \end{bmatrix}$, and so the deciphering matrix is $A^{-1} = \begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}$.

From Table 1 the numerical equivalent of the given ciphertext is

LN *GI* *HG* *YB* *VR* *EN* *JY*
12 14 7 9 8 7 25 2 22 18 5 14 10 25

QO
17 15

To obtain the plaintext pairs, we multiply each ciphertext vector by A^{-1}:

$$\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 12 \\ 14 \end{bmatrix} = \begin{bmatrix} 358 \\ 424 \end{bmatrix} = \begin{bmatrix} 20 \\ 8 \end{bmatrix} \quad \begin{matrix} T \\ H \end{matrix}$$
$$\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 213 \\ 259 \end{bmatrix} = \begin{bmatrix} 5 \\ 25 \end{bmatrix} \quad \begin{matrix} E \\ Y \end{matrix}$$
$$\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 227 \\ 250 \end{bmatrix} = \begin{bmatrix} 19 \\ 16 \end{bmatrix} \quad \begin{matrix} S \\ P \end{matrix}$$
$$\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 25 \\ 2 \end{bmatrix} = \begin{bmatrix} 610 \\ 503 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \end{bmatrix} \quad \begin{matrix} L \\ I \end{matrix} \pmod{26}$$
$$\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 22 \\ 18 \end{bmatrix} = \begin{bmatrix} 618 \\ 670 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix} \quad \begin{matrix} T \\ T \end{matrix}$$
$$\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 5 \\ 14 \end{bmatrix} = \begin{bmatrix} 190 \\ 291 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \quad \begin{matrix} H \\ E \end{matrix}$$
$$\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 10 \\ 25 \end{bmatrix} = \begin{bmatrix} 365 \\ 540 \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix} \quad \begin{matrix} A \\ T \end{matrix}$$
$$\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 17 \\ 15 \end{bmatrix} = \begin{bmatrix} 483 \\ 533 \end{bmatrix} = \begin{bmatrix} 15 \\ 13 \end{bmatrix} \quad \begin{matrix} O \\ M \end{matrix}$$

which yields the message *THEY SPLIT THE ATOM*.

6. Since we want a Hill 3-cipher, we will group the letters in triples. From Table 1 the numerical equivalents of the known plaintext are

I H A V E C O M E
9 8 1 22 5 3 15 13 5

and the numerical equivalent of the corresponding ciphertext are

H P A F Q G G D U
8 16 1 6 17 7 7 4 21

The corresponding plaintext and ciphertext vectors are

$$\mathbf{p}_1 = \begin{bmatrix} 9 \\ 8 \\ 1 \end{bmatrix} \leftrightarrow \mathbf{c}_1 = \begin{bmatrix} 8 \\ 16 \\ 1 \end{bmatrix}$$
$$\mathbf{p}_2 = \begin{bmatrix} 22 \\ 5 \\ 3 \end{bmatrix} \leftrightarrow \mathbf{c}_2 = \begin{bmatrix} 6 \\ 17 \\ 7 \end{bmatrix}$$
$$\mathbf{p}_3 = \begin{bmatrix} 15 \\ 13 \\ 5 \end{bmatrix} \leftrightarrow \mathbf{c}_3 = \begin{bmatrix} 7 \\ 4 \\ 21 \end{bmatrix}$$

We want to reduce $C = \begin{bmatrix} 8 & 16 & 1 \\ 6 & 17 & 7 \\ 7 & 4 & 21 \end{bmatrix}$ to I by elementary row operations and simultaneously apply these operations to $P = \begin{bmatrix} 9 & 8 & 1 \\ 22 & 5 & 3 \\ 15 & 13 & 5 \end{bmatrix}$.

The calculations are as follows:

$\left[\begin{array}{ccc|ccc} 8 & 16 & 1 & 9 & 8 & 1 \\ 6 & 17 & 7 & 22 & 5 & 3 \\ 7 & 4 & 21 & 15 & 13 & 5 \end{array}\right]$ Form the matrix $[C \mid P]$.

$\left[\begin{array}{ccc|ccc} 15 & 20 & 22 & 24 & 21 & 6 \\ 6 & 17 & 7 & 22 & 5 & 3 \\ 7 & 4 & 21 & 15 & 13 & 5 \end{array}\right]$ Add the third row to the first since 8^{-1} does not exist modulo 26.

$\left[\begin{array}{ccc|ccc} 1 & 140 & 154 & 168 & 147 & 42 \\ 6 & 17 & 7 & 22 & 5 & 3 \\ 7 & 4 & 21 & 15 & 13 & 5 \end{array}\right]$ Multiply the first row by $15^{-1} = 7 \pmod{26}$.

$$\left[\begin{array}{ccc|ccc} 1 & 10 & 24 & 12 & 17 & 16 \\ 6 & 17 & 7 & 22 & 5 & 3 \\ 7 & 4 & 21 & 15 & 13 & 5 \end{array}\right]$$

Replace the entries in the first row by their residues modulo 26.

$$\left[\begin{array}{ccc|ccc} 1 & 10 & 24 & 12 & 17 & 16 \\ 0 & -43 & -137 & -50 & -97 & -93 \\ 0 & -66 & -147 & -69 & -106 & -107 \end{array}\right]$$

Add −6 times the first row to the second and −7 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 10 & 24 & 12 & 17 & 16 \\ 0 & 9 & 19 & 2 & 7 & 11 \\ 0 & 12 & 9 & 9 & 24 & 23 \end{array}\right]$$

Replace the entries in the second and third rows by their residues modulo 26.

$$\left[\begin{array}{ccc|ccc} 1 & 10 & 24 & 12 & 17 & 16 \\ 0 & 1 & 57 & 6 & 21 & 33 \\ 0 & 12 & 9 & 9 & 24 & 23 \end{array}\right]$$

Multiply the second row by $9^{-1} = 3$ (modulo 26).

$$\left[\begin{array}{ccc|ccc} 1 & 10 & 24 & 12 & 17 & 16 \\ 0 & 1 & 5 & 6 & 21 & 7 \\ 0 & 12 & 9 & 9 & 24 & 23 \end{array}\right]$$

Replace the entries in the second row by their residues modulo 26.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -26 & -48 & -193 & -54 \\ 0 & 1 & 5 & 6 & 21 & 7 \\ 0 & 0 & -51 & -63 & -228 & -61 \end{array}\right]$$

Add −10 times the second row to the first and −12 times the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 15 & 24 \\ 0 & 1 & 5 & 6 & 21 & 7 \\ 0 & 0 & 1 & 15 & 6 & 17 \end{array}\right]$$

Replace the entries in the first and second row by their residues modulo 26.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 15 & 24 \\ 0 & 1 & 0 & -69 & -9 & -78 \\ 0 & 0 & 1 & 15 & 6 & 17 \end{array}\right]$$

Add −5 times the third row to the second.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 15 & 24 \\ 0 & 1 & 0 & 9 & 17 & 0 \\ 0 & 0 & 1 & 15 & 6 & 17 \end{array}\right]$$

Replace the entries in the second row by their residues modulo 26.

Thus, $(A^{-1})^T = \begin{bmatrix} 4 & 15 & 24 \\ 9 & 17 & 0 \\ 15 & 6 & 17 \end{bmatrix}$ and so the deciphering matrix is $A^{-1} = \begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}$.

From Table 1 the numerical equivalent of the given ciphertext is

H	*P*	*A*	*F*	*Q*	*G*	*G*	*D*	*U*	*G*	*G*	*D*
8	16	1	6	17	7	7	4	21	7	4	4

H	*P*	*G*	*O*	*D*	*Y*	*N*	*O*	*R*
8	16	7	15	4	25	14	15	18

To obtain the plaintext triples, we multiply each ciphertext vector by A^{-1}:

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 8 \\ 16 \\ 1 \end{bmatrix} = \begin{bmatrix} 191 \\ 398 \\ 209 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 1 \end{bmatrix} \quad \begin{matrix} I \\ H \\ A \end{matrix}$$

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 6 \\ 17 \\ 7 \end{bmatrix} = \begin{bmatrix} 282 \\ 421 \\ 263 \end{bmatrix} = \begin{bmatrix} 22 \\ 5 \\ 3 \end{bmatrix} \quad \begin{matrix} V \\ E \\ C \end{matrix}$$

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \\ 21 \end{bmatrix} = \begin{bmatrix} 379 \\ 299 \\ 525 \end{bmatrix} = \begin{bmatrix} 15 \\ 13 \\ 5 \end{bmatrix} \quad \begin{matrix} O \\ M \\ E \end{matrix}$$

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 124 \\ 197 \\ 236 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \\ 2 \end{bmatrix} \quad \begin{matrix} T \\ O \\ B \end{matrix}$$

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 8 \\ 16 \\ 7 \end{bmatrix} = \begin{bmatrix} 281 \\ 434 \\ 311 \end{bmatrix} = \begin{bmatrix} 21 \\ 18 \\ 25 \end{bmatrix} \quad \begin{matrix} U \\ R \\ Y \end{matrix}$$

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 15 \\ 4 \\ 25 \end{bmatrix} = \begin{bmatrix} 471 \\ 443 \\ 785 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \quad \begin{matrix} C \\ A \\ E \end{matrix}$$

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 14 \\ 15 \\ 18 \end{bmatrix} = \begin{bmatrix} 461 \\ 573 \\ 642 \end{bmatrix} = \begin{bmatrix} 19 \\ 1 \\ 18 \end{bmatrix} \quad \begin{matrix} S \\ A \\ R \end{matrix}$$

Finally, the message is *I HAVE COME TO BURY CAESAR*.

7. (a) Multiply each of the triples of the message by $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and reduce the results modulo 2.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The encoded message is 010110001.

(b) Reduce $[A \mid I]$ to $\left[I \mid A^{-1}\right]$ modulo 2.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$ Form the matrix $[A \mid I]$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 1 \end{array}\right]$$ Add the first row to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$ Replace 2 by its residue modulo 2.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$ Add the third row to the second row.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$ Replace 2 by its residue modulo 2.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$ Add the second row to the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$ Replace 2 by its residue modulo 2.

Thus $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The decoded message is 110101111, which is the original message.

8. Since 29 is a prime number, by Corollary 10.15.2 a matrix A with entries in Z_{29} is invertible if and only if $\det(A) \neq 0 \pmod{29}$.

Section 10.16

Exercise Set 10.16

1. Use induction on n, the case $n = 1$ being already given. If the result is true for $n - 1$, then

$$\begin{aligned} M^n &= M^{n-1}M \\ &= (PD^{n-1}P^{-1})(PDP^{-1}) \\ &= PD^{n-1}(P^{-1}P)DP^{-1} \\ &= PD^{n-1}DP^{-1} \\ &= PD^nP^{-1}, \end{aligned}$$

proving the result.

2. Using Table 1 and notations of Example 1, we derive the following equations:

$$a_n = \frac{1}{2}a_{n-1} + \frac{1}{4}b_{n-1}$$

$$b_n = \frac{1}{2}a_{n-1} + \frac{1}{2}b_{n-1} + \frac{1}{2}c_{n-1}$$

$$c_n = \frac{1}{4}b_{n-1} + \frac{1}{2}c_{n-1}.$$

The transition matrix is thus $M = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$.

The characteristic polynomial of M is

$$\det(\lambda I - M) = \lambda^3 - \left(\frac{3}{2}\right)\lambda^2 + \left(\frac{1}{2}\right)\lambda$$
$$= \lambda(\lambda-1)\left(\lambda - \frac{1}{2}\right),$$

so the eigenvalues of M are $\lambda = 1$, $\lambda_2 = \frac{1}{2}$, and $\lambda_3 = 0$. Corresponding eigenvectors (found by solving $(\lambda I - M)\mathbf{x} = 0$) are $\mathbf{e}_1 = [1 \quad 2 \quad 1]^T$, $\mathbf{e}_2 = [1 \quad 0 \quad -1]^T$, and $\mathbf{e}_3 = [1 \quad -2 \quad 1]^T$. Thus

$$M^n = PD^nP^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{1}{2}\right)^n & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

This yields

$$\mathbf{x}^{(n)} = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} \frac{1}{4}+\left(\frac{1}{2}\right)^{n+1} & \frac{1}{4} & \frac{1}{4}-\left(\frac{1}{2}\right)^{n+1} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4}-\left(\frac{1}{2}\right)^{n+1} & \frac{1}{4} & \frac{1}{4}+\left(\frac{1}{2}\right)^{n+1} \end{bmatrix}\begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}.$$

Remembering that $a_0 + b_0 + c_0 = 1$, we obtain

$$a_n = \frac{1}{4}a_0 + \frac{1}{4}b_0 + \frac{1}{4}c_0 + \left(\frac{1}{2}\right)^{n+1} a_0 - \left(\frac{1}{2}\right)^{n+1} c_0 = \frac{1}{4} + \left(\frac{1}{2}\right)^{n+1}(a_0 - c_0)$$

$$b_n = \frac{1}{2}a_0 + \frac{1}{2}b_0 + \frac{1}{2}c_0 = \frac{1}{2}$$

$$c_n = \frac{1}{4}a_0 + \frac{1}{4}b_0 + \frac{1}{2}c_0 - \left(\frac{1}{2}\right)^{n+1} a_0 + \left(\frac{1}{2}\right)^{n+1} c_0 = \frac{1}{4} - \left(\frac{1}{2}\right)^{n+1}(a_0 - c_0).$$

Since $\left(\frac{1}{2}\right)^{n+1}$ approaches zero as $n \to \infty$, we obtain $a_n \to \frac{1}{4}$, $b_n \to \frac{1}{2}$, and $c_n \to \frac{1}{4}$ as $n \to \infty$.

3. Call M_1 the matrix of Example 1, and M_2 the matrix of Exercise 2. Then $\mathbf{x}^{(2n)} = (M_2M_1)^n \mathbf{x}^{(0)}$ and $\mathbf{x}^{(2n+1)} = M_1(M_2M_1)^n \mathbf{x}^{(0)}$. We have

$$M_2M_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{8} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{8} & \frac{1}{4} \end{bmatrix}.$$

The characteristic polynomial of this matrix is $\lambda^3 - \frac{5}{4}\lambda^2 + \frac{1}{4}\lambda$, so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = \frac{1}{4}$, $\lambda_3 = 0$. Corresponding eigenvectors are $\mathbf{e}_1 = [5 \quad 6 \quad 1]^T$, $\mathbf{e}_2 = [-1 \quad 0 \quad 1]^T$, and $\mathbf{e}_3 = [1 \quad -2 \quad 1]^T$. Thus,

$$(M_2M_1)^n = PD^nP^{-1} = \begin{bmatrix} 5 & -1 & 1 \\ 6 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{1}{4}\right)^n & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ -\frac{1}{3} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Using the notation of Example 1 (recall that $a_0 + b_0 + c_0 = 1$), we obtain

$$a_{2n} = \frac{5}{12} + \frac{1}{6\cdot 4^n}(2a_0 - b_0 - 4c_0)$$
$$b_{2n} = \frac{1}{2}$$
$$c_{2n} = \frac{1}{12} - \frac{1}{6\cdot 4^n}(2a_0 - b_0 - 4c_0)$$

and

$$a_{2n+1} = \frac{2}{3} + \frac{1}{6\cdot 4^n}(2a_0 - b_0 - 4c_0)$$
$$b_{2n+1} = \frac{1}{3} - \frac{1}{6\cdot 4^n}(2a_0 - b_0 - 4c_0)$$
$$c_{2n+1} = 0.$$

4. The characteristic polynomial of M is $(\lambda-1)\left(\lambda - \frac{1}{2}\right)$, so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$. Corresponding eigenvectors are easily found to be $\mathbf{e}_1 = [1 \quad 0]^T$ and $\mathbf{e}_2 = [1 \quad -1]^T$. From this point, the verification of Equation (7) is in the text.

5. From Equation (9), if $b_0 = .25 = \frac{1}{4}$, we get $b_1 = \frac{\frac{1}{4}}{\frac{9}{8}} = \frac{2}{9}$, then $b_2 = \frac{\frac{2}{9}}{\frac{10}{9}} = \frac{1}{5}$, $b_3 = \frac{\frac{1}{5}}{\frac{11}{10}} = \frac{2}{11}$, and, in general, $b_n = \frac{2}{8+n}$. We will reach $\frac{2}{20} = .10$ in 12 generations. According to Equation (8), under the controlled program the percentage would be $\frac{1}{2^{14}}$ in 12 generations, or $\frac{1}{16,384} = .00006 = .006\%$.

6. $$P^{-1}\mathbf{x}^{(0)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{12}(1+\sqrt{5}) \\ \frac{1}{12}(1-\sqrt{5}) \end{bmatrix}$$

$$D^n P^{-1}\mathbf{x}^{(0)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{3}\left[\frac{1}{4}(1+\sqrt{5})\right]^{n+1} \\ \frac{1}{3}\left[\frac{1}{4}(1-\sqrt{5})\right]^{n+1} \end{bmatrix}$$

$$PD^n P^{-1}\mathbf{x}^{(0)} = \begin{bmatrix} \frac{1}{2}+\frac{1}{3}\cdot\frac{1}{4^{n+2}}\left[(-3-\sqrt{5})(1+\sqrt{5})^{n+1}+(-3+\sqrt{5})(1-\sqrt{5})^{n+1}\right] \\ \frac{1}{3}\cdot\frac{1}{4^{n+1}}\left[(1+\sqrt{5})^{n+1}+(1-\sqrt{5})^{n+1}\right] \\ \frac{1}{3}\cdot\frac{1}{4^{n+1}}\left[(1+\sqrt{5})^{n}+(1-\sqrt{5})^{n}\right] \\ \frac{1}{3}\cdot\frac{1}{4^{n+1}}\left[(1+\sqrt{5})^{n}+(1-\sqrt{5})^{n}\right] \\ \frac{1}{3}\cdot\frac{1}{4^{n+1}}\left[(1+\sqrt{5})^{n+1}+(1-\sqrt{5})^{n+1}\right] \\ \frac{1}{2}+\frac{1}{3}\cdot\frac{1}{4^{n+2}}\left[(-3-\sqrt{5})(1+\sqrt{5})^{n+1}+(-3+\sqrt{5})(1-\sqrt{5})^{n+1}\right] \end{bmatrix}$$

[As n tends to infinity, $\frac{1}{4^{n+2}}$ and $\frac{1}{4^{n+1}}$ approach 0, so $\mathbf{x}^{(n)}$ approaches $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$.

7. From (13) we have that the probability that the limiting sibling-pairs will be type (A, AA) is
$a_0 + \frac{2}{3}b_0 + \frac{1}{3}c_0 + \frac{2}{3}d_0 + \frac{1}{3}e_0$.
The proportion of A genes in the population at the outset is as follows: all the type

(*A*, *AA*) genes, $\frac{2}{3}$ of the type (*A*, *Aa*) genes, $\frac{1}{3}$ the type (*A*, *aa*) genes, etc. ...yielding $a_0+\frac{2}{3}b_0+\frac{1}{3}c_0+\frac{2}{3}d_0+\frac{1}{3}e_0.$

8. From an (*A*, *AA*) pair we get only (*A*, *AA*) pairs and similarly for (*a*, *aa*). From either (*A*, *aa*) or (*a*, *AA*) pairs we must get an *Aa* female, who will not mature. Thus no offspring will come from such pairs. The transition matrix is then
$$\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}.$$

9. For the first column of *M* we realize that parents of type (*A*, *AA*) can produce offspring only of that type, and similarly for the last column. The fifth column is like the second column, and follows the analysis in the text. For the middle two columns, say the third, note that male offspring from (*A*, *aa*) must be of type *a*, and females are of type *Aa*, because of the way the genes are inherited.

Section 10.17

Exercise Set 10.17

1. (a) The characteristic polynomial of *L* is $\lambda^2-\lambda-\frac{3}{4}$, so the eigenvalues of *L* are $\lambda=\frac{3}{2}$ and $\lambda=-\frac{1}{2}$, thus $\lambda_1=\frac{3}{2}$ and $\mathbf{x}_1=\begin{bmatrix}1\\\frac{1}{2}\\\frac{3}{2}\end{bmatrix}=\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}$ is the corresponding eigenvector.

(b) $\mathbf{x}^{(1)}=L\mathbf{x}^{(0)}=\begin{bmatrix}100\\50\end{bmatrix}$, $\mathbf{x}^{(2)}=L\mathbf{x}^{(1)}=\begin{bmatrix}175\\50\end{bmatrix}$,
$\mathbf{x}^{(3)}=L\mathbf{x}^{(2)}=\begin{bmatrix}250\\88\end{bmatrix}$,
$\mathbf{x}^{(4)}=L\mathbf{x}^{(3)}=\begin{bmatrix}382\\125\end{bmatrix}$, $\mathbf{x}^{5}=L\mathbf{x}^{(4)}=\begin{bmatrix}570\\191\end{bmatrix}$

(c) $\mathbf{x}^{(6)}=L\mathbf{x}^{(5)}=\begin{bmatrix}857\\285\end{bmatrix}$
$\mathbf{x}^{(6)}\simeq\lambda_1\mathbf{x}^{(5)}=\begin{bmatrix}855\\287\end{bmatrix}$

5. a_1 is the average number of offspring produced in the first age period. a_2b_1 is the number of offspring produced in the second period times the probability that the female will live into the second period, i.e., it is the expected number of offspring per female during the second period, and so on for all the periods. Thus, the sum of these, which is the net reproduction rate, is the expected number of offspring produced by a given female during her expected lifetime.

7. $R=0+4\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)=\frac{19}{8}=2.375$

8. $$R=0+(.00024)(.99651)+\cdots+(.00240)(.99651)\cdots(.987)=1.49611.$$

Section 10.18

Exercise Set 10.18

1. (a) The characteristic polynomial of *L* is $\lambda^3-2\lambda-\frac{3}{8}=\left(\lambda-\frac{3}{2}\right)\left[\lambda^2+\left(\frac{3}{2}\right)\lambda+\frac{1}{4}\right]$, so $\lambda_1=\frac{3}{2}$. Thus *h*, the fraction harvested of each age group, is $1-\frac{2}{3}=\frac{1}{3}$ so the yield is $33\frac{1}{3}\%$ of the population. The eigenvector corresponding to $\lambda=\frac{3}{2}$ is $\begin{bmatrix}1\\\frac{1}{3}\\\frac{1}{18}\end{bmatrix}$; this is the age distribution vector after each harvest.

(b) From Equation (10), the age distribution vector $\mathbf{x}_1$ is $\begin{bmatrix}1&\frac{1}{2}&\frac{1}{8}\end{bmatrix}^T$. Equation (9) tells us that $h_1=1-\frac{1}{\frac{19}{8}}=\frac{11}{19}$, so we harvest $\frac{11}{19}$ or 57.9% of the youngest age class. Since $L\mathbf{x}_1=\begin{bmatrix}\frac{19}{8}&\frac{1}{2}&\frac{1}{8}\end{bmatrix}^T$, the youngest class contains 79.2% of the population. Thus the yield is 57.9% of 79.2%, or 45.8% of the population.

2. The Leslie matrix of Example 1 has $b_1 = .845,\ b_2 = .975,\ b_3 = .965,$ etc. This, together with the harvesting data from Equations (13) and the formula of Equation (5) yields

$\mathbf{x}_1 = [1\ \ .845\ \ .824\ \ .795\ \ .755\ \ .699\ \ .626\ \ .5323\ \ 0\ \ 0\ \ 0\ \ 0]^T$

$L\mathbf{x}_1 = [2.090\ \ .845\ \ .824\ \ .795\ \ .755\ \ .699\ \ .626\ \ .532\ \ .418\ \ 0\ \ 0\ \ 0]^T.$

The total of the entries of $L\mathbf{x}_1$ is 7.584. The proportion of sheep harvested is $h_1(L\mathbf{x}_1)_2 + h_9(L\mathbf{x}_1)_9 = 1.51,$ or 19.9% of the population.

4. In this situation we have $h_I \neq 0,$ and $h_1 = h_2 = \cdots = h_{I-1} = h_{I+1} = \cdots = h_n = 0.$ Equation (4) then takes the form

$a_1 + a_2b_1 + a_3b_1b_2 + \cdots + a_Ib_1b_2\cdots b_{I-1}(1-h_I) + a_{I+1}b_1b_2\cdots b_I(1-h_I) + \cdots + a_nb_1b_2\cdots b_{n-1}(1-h_I) = 1.$

$(1-h_I)[a_Ib_1b_2\cdots b_{I-1} + a_{I+1}b_1b_2\cdots b_I + \cdots + a_nb_1b_2\cdots b_{n-1}] = 1 - a_1 - a_2b_1 - \cdots - a_{I-1}b_1b_2\cdots b_{I-2}.$

So
$$\begin{aligned} h_I &= \frac{a_1 + a_2b_1 + \cdots + a_{I-1}b_1b_2\cdots b_{I-2} - 1}{a_Ib_1b_2\cdots b_{I-1} + \cdots + a_nb_1b_2\cdots b_{n-1}} + 1 \\ &= \frac{a_1 + a_2b_1 + \cdots + a_{I-1}b_1b_2\cdots b_{I-2} - 1 + a_Ib_1b_2\cdots b_{I-1} + \cdots + a_nb_1b_2\cdots b_{n-1}}{a_Ib_1b_2\cdots b_{I-1} + \cdots + a_nb_1b_2\cdots b_{n-1}} \\ &= \frac{R-1}{a_Ib_1b_2\cdots b_{I-1} + \cdots + a_nb_1b_2\cdots b_{n-1}} \end{aligned}$$

5. Here $h_J = 1,\ h_I \neq 0,$ and all the other h_k's are zero. Then Equation (4) becomes

$a_1 + a_2b_1 + \cdots + a_{I-1}b_1b_2\cdots b_{I-2} + (1-h_I)[a_Ib_1b_2\cdots b_{I-1} + \cdots + a_{J-1}b_1b_2\cdots b_{J-2}] = 1.$

We solve for h_I to obtain
$$\begin{aligned} h_I &= \frac{a_1 + a_2b_1 + \cdots + a_{I-1}b_1b_2\cdots b_{I-2} - 1}{a_Ib_1b_2\cdots b_{I-1} + \cdots + a_{J-1}b_1b_2\cdots b_{J-2}} + 1 \\ &= \frac{a_1 + a_2b_1 + \cdots + a_{J-1}b_1b_2\cdots b_{J-2} - 1}{a_Ib_1b_2\cdots b_{I-1} + \cdots + a_{J-1}b_1b_2\cdots b_{J-2}}. \end{aligned}$$

Section 10.19

Exercise Set 10.19

1. From Theorem 10.19.1, we compute $a_0 = \frac{1}{\pi}\int_0^{2\pi}(t-\pi)^2\,dt = \frac{2}{3}\pi^2,\ a_k = \frac{1}{\pi}\int_0^{2\pi}(t-\pi)^2\cos kt\,dt = \frac{4}{k^2},$ and

$b_k = \frac{1}{\pi}\int_0^{2\pi}(t-\pi)^2\sin kt\,dt = 0.$ So the least-squares trigonometric polynomial of order 3 is

$$\frac{\pi^2}{3} + 4\cos t + \cos 2t + \frac{4}{9}\cos 3t.$$

2. From Theorem 10.19.2, we compute $a_0 = \frac{2}{T}\int_0^T t^2\,dt = \frac{2}{3}T^2,\ a_k = \frac{2}{T}\int_0^T t^2\cos\frac{2k\pi t}{T}\,dt = \frac{T^2}{k^2\pi^2}$ and

$b_k = \frac{2}{T}\int_0^T t^2\sin\frac{2k\pi t}{T}\,dt = -\frac{T^2}{k\pi}.$

So the least-squares trigonometric polynomial of order 4 is

$$\frac{T^2}{3} + \frac{T^2}{\pi^2}\left(\cos\frac{2\pi t}{T} + \frac{1}{4}\cos\frac{4\pi t}{T} + \frac{1}{9}\cos\frac{6\pi t}{T} + \frac{1}{16}\cos\frac{8\pi t}{T}\right) - \frac{T^2}{\pi}\left(\sin\frac{2\pi t}{T} + \frac{1}{2}\sin\frac{4\pi t}{T} + \frac{1}{3}\sin\frac{6\pi t}{T} + \frac{1}{4}\sin\frac{8\pi t}{T}\right).$$

3. From Theorem 10.19.2, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)dt = \frac{1}{\pi}\int_0^{\pi} \sin t\, dt = \frac{2}{\pi}$. (Note the upper limit on the second integral),

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_0^{\pi} \sin t \cos kt\, dt \\ &= \frac{1}{\pi}\left(\frac{1}{k^2-1}[k \sin kt \sin t + \cos kt \cos t\right)\Bigg|_0^{\pi} \\ &= \frac{1}{\pi}\left(\frac{1}{k^2-1}[0+(-1)^{k-1}-1]\right) \\ &= \begin{cases} 0 & \text{if } k \text{ is odd} \\ -\dfrac{2}{\pi(k^2-1)} & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

$$b_k = \frac{1}{\pi}\int_0^{\pi} \sin kt \sin t\, dt = \begin{cases} \frac{1}{2} & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}.$$

So the least-squares trigonometric polynomial of order 4 is $\frac{1}{\pi} + \frac{1}{2}\sin t - \frac{2}{3\pi}\cos 2t - \frac{2}{15\pi}\cos 4t$.

4. From Theorem 10.19.2, $a_0 = \frac{1}{\pi}\int_0^{2\pi} \sin\frac{1}{2}t\, dt = \frac{4}{\pi}$.

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_0^{2\pi} \sin\frac{1}{2}t \cos kt\, dt \\ &= -\frac{4}{\pi(4k^2-1)} \\ &= -\frac{4}{\pi(2k-1)(2k+1)}, \end{aligned}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} \sin\frac{1}{2}t \sin kt\, dt = 0.$$

So the least-square trigonometric polynomial of order n is $\frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos t}{1\cdot 3} + \frac{\cos 2t}{3\cdot 5} + \frac{\cos 3t}{5\cdot 7} + \cdots + \frac{\cos nt}{(2n-1)(2n+1)}\right)$.

5. From Theorem 10.19.2, $a_0 = \frac{2}{T}\int_0^{T} f(t)dt = \frac{2}{T}\int_0^{\frac{1}{2}T} t\, dt + \frac{2}{T}\int_{\frac{1}{2}T}^{T} (T-t)dt = \frac{T}{2}$.

$$\begin{aligned} a_k &= \frac{2}{T}\int_0^{\frac{1}{2}T} t \cos\frac{2k\pi t}{T} dt + \frac{2}{T}\int_{\frac{1}{2}T}^{T} (T-t)\cos\frac{2k\pi t}{T} dt \\ &= \frac{4T}{4k^2\pi^2}((-1)^k - 1) \\ &= \begin{cases} 0 & \text{if } k \text{ is even} \\ \dfrac{8T}{(2k)^2\pi^2} & \text{if } k \text{ is odd} \end{cases}. \end{aligned}$$

So the least-squares trigonometric polynomial of order n is:

$$\frac{T}{4} - \frac{8T}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi t}{T} + \frac{1}{6^2}\cos\frac{6\pi t}{T} + \frac{1}{10^2}\cos\frac{10\pi t}{T} + \cdots + \frac{1}{(2n)^2}\cos\frac{2\pi n t}{T}\right)$$

if n is even; the last term involves $n - 1$ if n is odd.

6. (a) $\|1\| = \sqrt{\int_0^{2\pi} dt} = \sqrt{2\pi}$

(b) $\|\cos kt\| = \sqrt{\int_0^{2\pi} \cos^2 kt\, dt}$
$= \sqrt{\int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right) dt}$
$= \sqrt{\pi}.$

(c) $\|\sin kt\| = \sqrt{\int_0^{2\pi} \sin^2 kt\, dt}$
$= \sqrt{\int_0^{2\pi} \left(\frac{1-\cos 2t}{2}\right) dt}$
$= \sqrt{\pi}$

Section 10.20

Exercise Set 10.20

1. (a) Equation (2) is $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}3\\5\end{bmatrix} + c_3\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix}$ and Equation (3) is $c_1 + c_2 + c_3 = 1.$ These equations can be written in combined matrix form as $\begin{bmatrix}1 & 3 & 4\\1 & 5 & 2\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}3\\3\\1\end{bmatrix}.$

This system has the unique solution $c_1 = \frac{1}{5},$ $c_2 = \frac{2}{5},$ and $c_3 = \frac{2}{5}.$ Because these coefficients are all nonnegative, it follows that **v** is a convex combination of the vectors $\mathbf{v}_1,$ $\mathbf{v}_2,$ and $\mathbf{v}_3.$

(b) As in part (a) the system for $c_1,$ $c_2,$ and c_3 is $\begin{bmatrix}1 & 3 & 4\\1 & 5 & 2\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}2\\4\\1\end{bmatrix}$ which has the unique solution $c_1 = \frac{2}{5},$ $c_2 = \frac{4}{5},$ and $c_3 = -\frac{1}{5}.$ Because one of these coefficients is negative, it follows that **v** is not a convex combination of the vectors $\mathbf{v}_1,$ $\mathbf{v}_2,$ and $\mathbf{v}_3.$

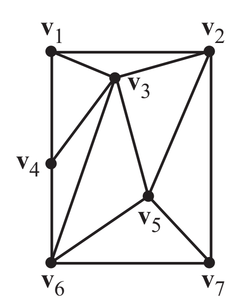
(c) As in part (a) the system for $c_1,$ $c_2,$ and c_3 is $\begin{bmatrix}3 & -2 & 3\\3 & -2 & 0\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ which has the unique solution $c_1 = \frac{2}{5},$ $c_2 = \frac{3}{5},$ and $c_3 = 0.$ Because these coefficients are all nonnegative, it follows that **v** is a convex combination of the vectors $\mathbf{v}_1,$ $\mathbf{v}_2,$ and $\mathbf{v}_3.$

(d) As in part (a) the system for $c_1,$ $c_2,$ and c_3 is $\begin{bmatrix}3 & -2 & 3\\3 & -2 & 0\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$ which has the unique solution $c_1 = \frac{4}{15},$ $c_2 = \frac{6}{15},$ and $c_3 = \frac{5}{15}.$ Because these coefficients are all nonnegative, it follows that **v** is a convex combination of the vectors $\mathbf{v}_1,$ $\mathbf{v}_2,$ and $\mathbf{v}_3.$

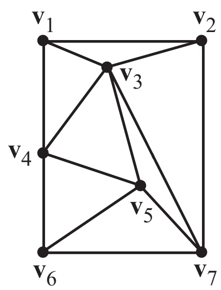
2. For both triangulations the number of triangles, *m*, is equal to 7; the number of vertex points, *n*, is equal to 7; and the number of boundary vertex points, *k*, is equal to 5. Equation (7), $m = 2n - 2 - k$, becomes $7 = 2(7) - 2 - 5,$ or $7 = 7.$

3. Combining everything that is given in the statement of the problem, we obtain:
$$\begin{aligned}\mathbf{w} &= M\mathbf{v} + \mathbf{b}\\ &= M(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) + (c_1 + c_2 + c_3)\mathbf{b}\\ &= c_1(M\mathbf{v}_1 + \mathbf{b}) + c_2(M\mathbf{v}_2 + \mathbf{b}) + c_3(M\mathbf{v}_3 + \mathbf{b})\\ &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3.\end{aligned}$$

4. (a)



(b)



5. (a) Let $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Then the three matrix equations $M\mathbf{v}_i + \mathbf{b} = \mathbf{w}_i$, $i = 1, 2, 3$, can be written as the six scalar equations

$$m_{11} + m_{12} + b_1 = 4$$
$$m_{21} + m_{22} + b_2 = 3$$

$$2m_{11} + 3m_{12} + b_1 = 9$$
$$2m_{21} + 3m_{22} + b_2 = 5$$

$$2m_{11} + m_{12} + b_1 = 5$$
$$2m_{21} + m_{22} + b_2 = 3$$

The first, third, and fifth equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} m_{11} \\ m_{12} \\ b_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$ and the second, fourth, and sixth equations as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} m_{21} \\ m_{22} \\ b_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}.$$

The first system has the solution $m_{11} = 1$, $m_{12} = 2$, $b_1 = 1$ and the second system has the solution $m_{21} = 0$, $m_{22} = 1$, $b_2 = 2$.

Thus we obtain $M = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) As in part (a), we are led to the following two linear systems:

$$\begin{bmatrix} -2 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} m_{11} \\ m_{12} \\ b_1 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 5 \end{bmatrix}$$ and

$$\begin{bmatrix} -2 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} m_{21} \\ m_{22} \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}.$$

Solving these two linear systems leads to $M = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) As in part (a), we are led to the following two linear systems:

$$\begin{bmatrix} -2 & 1 & 1 \\ 3 & 5 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} m_{11} \\ m_{12} \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$ and

$$\begin{bmatrix} -2 & 1 & 1 \\ 3 & 5 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} m_{21} \\ m_{22} \\ b_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}.$$

Solving these two linear systems leads to $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

(d) As in part (a), we are led to the following two linear systems:

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 1 \\ -4 & -2 & 1 \end{bmatrix}\begin{bmatrix} m_{11} \\ m_{12} \\ b_1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{7}{2} \\ -\frac{7}{2} \end{bmatrix}$$ and

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 1 \\ -4 & -2 & 1 \end{bmatrix}\begin{bmatrix} m_{21} \\ m_{22} \\ b_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -9 \end{bmatrix}.$$

Solving these two linear systems leads to $M = \begin{bmatrix} \frac{1}{2} & 1 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}$.

7. (a) The vertices $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ of a triangle can be written as the convex combinations $\mathbf{v}_1 = (1)\mathbf{v}_1 + (0)\mathbf{v}_2 + (0)\mathbf{v}_3$, $\mathbf{v}_2 = (0)\mathbf{v}_1 + (1)\mathbf{v}_2 + (0)\mathbf{v}_3$, and $\mathbf{v}_3 = (0)\mathbf{v}_1 + (0)\mathbf{v}_2 + (1)\mathbf{v}_3$. In each of these cases, precisely two of the coefficients are zero and one coefficient is one.

(b) If, for example, $\mathbf{v}$ lies on the side of the triangle determined by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ then from Exercise 6(a) we must have that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + (0)\mathbf{v}_3$ where $c_1 + c_2 = 1$. Thus at least one of the coefficients, in this example c_3, must equal zero.

(c) From part (b), if at least one of the coefficients in the convex combination is zero, then the vector must lie on one of the sides of the triangle. Consequently, none of the coefficients can be zero if the vector lies in the interior of the triangle.

8. (a) Consider the vertex $\mathbf{v}_1$ of the triangle and its opposite side determined by the vectors $\mathbf{v}_2$ and $\mathbf{v}_3$. The midpoint $\mathbf{v}_m$ of this opposite side is $\frac{(\mathbf{v}_2+\mathbf{v}_3)}{2}$ and the point on the line segment from $\mathbf{v}_1$ to $\mathbf{v}_m$ that is two-thirds of the distance to $\mathbf{v}_m$ is given by

$$\begin{aligned}\frac{1}{3}\mathbf{v}_1+\frac{2}{3}\mathbf{v}_m &= \frac{1}{3}\mathbf{v}_1+\frac{2}{3}\left(\frac{\mathbf{v}_2+\mathbf{v}_3}{2}\right)\\ &= \frac{1}{3}\mathbf{v}_1+\frac{1}{3}\mathbf{v}_2+\frac{1}{3}\mathbf{v}_3.\end{aligned}$$

(b) $\frac{1}{3}\begin{bmatrix}2\\3\end{bmatrix}+\frac{1}{3}\begin{bmatrix}5\\2\end{bmatrix}+\frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}8\\6\end{bmatrix}=\begin{bmatrix}\frac{8}{3}\\2\end{bmatrix}$

Chapter 2

Determinants

Section 2.1

Exercise Set 2.1

1. $M_{11} = \begin{vmatrix} 7 & -1 \\ 1 & 4 \end{vmatrix} = 28-(-1) = 29$

$C_{11} = (-1)^{1+1}M_{11} = 29$

$M_{12} = \begin{vmatrix} 6 & -1 \\ -3 & 4 \end{vmatrix} = 24-3 = 21$

$C_{12} = (-1)^{1+2}M_{12} = -21$

$M_{13} = \begin{vmatrix} 6 & 7 \\ -3 & 1 \end{vmatrix} = 6-(-21) = 27$

$C_{13} = (-1)^{1+3}M_{13} = 27$

$M_{21} = \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} = -8-3 = -11$

$C_{21} = (-1)^{2+1}M_{21} = 11$

$M_{22} = \begin{vmatrix} 1 & 3 \\ -3 & 4 \end{vmatrix} = 4-(-9) = 13$

$C_{22} = (-1)^{2+2}M_{22} = 13$

$M_{23} = \begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix} = 1-6 = -5$

$C_{23} = (-1)^{2+3}M_{23} = 5$

$M_{31} = \begin{vmatrix} -2 & 3 \\ 7 & -1 \end{vmatrix} = 2-21 = -19$

$C_{31} = (-1)^{3+1}M_{31} = -19$

$M_{32} = \begin{vmatrix} 1 & 3 \\ 6 & -1 \end{vmatrix} = -1-18 = -19$

$C_{32} = (-1)^{3+2}M_{32} = 19$

$M_{33} = \begin{vmatrix} 1 & -2 \\ 6 & 7 \end{vmatrix} = 7-(-12) = 19$

$C_{33} = (-1)^{3+3}M_{33} = 19$

3. **(a)** $M_{13} = \begin{vmatrix} 0 & 0 & 3 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix} = 3(4-4) = 0$

$C_{13} = (-1)^{1+3}M_{13} = 0$

(b) $M_{23} = \begin{vmatrix} 4 & -1 & 6 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix}$

$$\begin{aligned} &= 4\begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 6\begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix} \\ &= 4(2-14)+(8-56)+6(4-4) \\ &= -96 \end{aligned}$$

$C_{23} = (-1)^{2+3}M_{23} = 96$

(c) $M_{22} = \begin{vmatrix} 4 & 1 & 6 \\ 4 & 0 & 14 \\ 4 & 3 & 2 \end{vmatrix}$

$$\begin{aligned} &= -4\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} 4 & 6 \\ 4 & 2 \end{vmatrix} - 14\begin{vmatrix} 4 & 1 \\ 4 & 3 \end{vmatrix} \\ &= -4(2-18)-14(12-4) \\ &= -48 \end{aligned}$$

$C_{22} = (-1)^{2+2}M_{22} = -48$

(d) $M_{21} = \begin{vmatrix} -1 & 1 & 6 \\ 1 & 0 & 14 \\ 1 & 3 & 2 \end{vmatrix}$

$$\begin{aligned} &= -1\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} -1 & 6 \\ 1 & 2 \end{vmatrix} - 14\begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} \\ &= -(2-18)-14(-3-1) \\ &= 72 \end{aligned}$$

$C_{21} = (-1)^{2+1}M_{21} = -72$

5. $\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = 12-(-10) = 22$

$$\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}^{-1} = \frac{1}{22}\begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{11} & -\frac{5}{22} \\ \frac{1}{11} & \frac{3}{22} \end{bmatrix}$$

7. $\begin{vmatrix} -5 & 7 \\ -7 & -2 \end{vmatrix} = 10-(-49) = 59$

$$\begin{bmatrix} -5 & 7 \\ -7 & -2 \end{bmatrix}^{-1} = \frac{1}{59}\begin{bmatrix} -2 & -7 \\ 7 & -5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{59} & -\frac{7}{59} \\ \frac{7}{59} & -\frac{5}{59} \end{bmatrix}$$

9. $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix} = (a-3)(a-2)-(5)(-3)$
$$= a^2-5a+6+15$$
$$= a^2-5a+21$$

11. $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix}\begin{matrix} -2 & 1 \\ 3 & 5 \\ 1 & 6 \end{matrix}$
$$= [(-2)(5)(2)+(1)(-7)(1)+(4)(3)(6)]-[(4)(5)(1)+(-2)(-7)(6)+(1)(3)(2)]$$
$$= [-20-7+72]-[20+84+6]$$
$$= -65$$

13. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix}\begin{matrix} 3 & 0 \\ 2 & -1 \\ 1 & 9 \end{matrix}$
$$= [(3)(-1)(-4)+(0)(5)(1)+(0)(2)(9)]-[0(-1)(1)+(3)(5)(9)+(0)(2)(-4)]$$
$$= 12-135$$
$$= -123$$

15. $\det(A) = (\lambda-2)(\lambda+4)-(-5)$
$$= \lambda^2+2\lambda-8+5$$
$$= \lambda^2+2\lambda-3$$
$$= (\lambda-1)(\lambda+3)$$
$\det(A) = 0$ for $\lambda = 1$ or -3.

17. $\det(A) = (\lambda-1)(\lambda+1)-0 = (\lambda-1)(\lambda+1)$
$\det(A) = 0$ for $\lambda = 1$ or -1.

19. **(a)** $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix} = 3\begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 0\begin{vmatrix} 2 & 5 \\ 1 & -4 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ 1 & 9 \end{vmatrix}$
$$= 3(4-45)$$
$$= -123$$

(b) $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix} = 3\begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 9 & -4 \end{vmatrix} + 1\begin{vmatrix} 0 & 0 \\ -1 & 5 \end{vmatrix}$
$$= 3(4-45)-2(0-0)+(0-0)$$
$$= -123$$

(c) $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix} = -2\begin{vmatrix} 0 & 0 \\ 9 & -4 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} - 5\begin{vmatrix} 3 & 0 \\ 1 & 9 \end{vmatrix}$
$$= -2(0-0)-(-12-0)-5(27-0)$$
$$= 12-135$$
$$= -123$$

(d) $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix} = -0\begin{vmatrix} 2 & 5 \\ 1 & -4 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} - 9\begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix}$

$$\begin{aligned} &= -(-12-0) - 9(15-0) \\ &= 12 - 135 \\ &= -123 \end{aligned}$$

(e) $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix} = 1\begin{vmatrix} 0 & 0 \\ -1 & 5 \end{vmatrix} - 9\begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix}$

$$\begin{aligned} &= (0-0) - 9(15-0) - 4(-3-0) \\ &= -135 + 12 \\ &= -123 \end{aligned}$$

(f) $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix} = 0\begin{vmatrix} 2 & -1 \\ 1 & 9 \end{vmatrix} - 5\begin{vmatrix} 3 & 0 \\ 1 & 9 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix}$

$$\begin{aligned} &= -5(27-0) - 4(-3-0) \\ &= -135 + 12 \\ &= -123 \end{aligned}$$

21. Expand along the second column.

$$\det(A) = 5\begin{vmatrix} -3 & 7 \\ -1 & 5 \end{vmatrix} = 5[-15-(-7)] = -40$$

23. Expand along the first column.

$$\begin{aligned} \det(A) &= 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} - 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} + 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} \\ &= (k^3-k^3) - (k^3-k^3) + (k^3-k^3) \\ &= 0 \end{aligned}$$

25. Expand along the third column, then along the first rows of the 3×3 matrices.

$$\begin{aligned} \det(A) &= -3\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix} \\ &= 3\left(3\begin{vmatrix} 2 & -2 \\ 10 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & 2 \\ 2 & 10 \end{vmatrix}\right) - 3\left(3\begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} 2 & -2 \\ 4 & 0 \end{vmatrix} + 5\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix}\right) \\ &= -3[3(24) - 3(8) + 5(16)] - 3[3(2) - 3(8) + 5(-6)] \\ &= -3(128) - 3(-48) \\ &= -240 \end{aligned}$$

27. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1)(-1)(1) = -1$

29. $\begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 1 & 2 & 3 & 8 \end{vmatrix} = (0)(2)(3)(8) = 0$

31. $\begin{vmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{vmatrix} = (1)(1)(2)(3) = 6$

33. Expand along the third column.

$$\begin{aligned} \begin{vmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ \sin(\theta)-\cos(\theta) & \sin(\theta)+\cos(\theta) & 1 \end{vmatrix} &= \begin{vmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{vmatrix} \\ &= \sin^2(\theta) - (-\cos^2(\theta)) \\ &= \sin^2(\theta) + \cos^2(\theta) \\ &= 1 \end{aligned}$$

35. $M_{11} = \begin{vmatrix} 1 & f \\ 0 & 1 \end{vmatrix} = 1$ in both matrices, so expanding along the first row or column gives that $d_2 = d_1 + \lambda$.

True/False 2.1

(a) False; the determinant is $ad - bc$.

(b) False; $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

(c) True; if $i + j$ is even then $(-a)^{i+j} = 1$ and $C_{ij} = (-1)^{i+j} M_{ij} = M_{ij}$. Similarly, if $C_{ij} = (-1)^{i+j} M_{ij} = M_{ij}$, then $(-1)^{i+j} = 1$ so $i + j$ must be even.

(d) True; let $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$, then $C_{12} = (-1)\begin{vmatrix} b & e \\ c & f \end{vmatrix} = -(bf - ce)$ and $C_{21} = (-1)\begin{vmatrix} b & c \\ e & f \end{vmatrix} = -(bf - ce)$. Similar arguments show that $C_{23} = C_{32}$ and $C_{13} = C_{31}$.

(e) True; the value is det(A) regardless of the row or column chosen.

(f) True

(g) False; the determinant would be the product, not the sum.

(h) False; let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then det($A$) = 1, but $\det(cA) = \begin{vmatrix} c & 0 \\ 0 & c \end{vmatrix} = c^2$. In general, if A is an $n \times n$ matrix, then $\det(cA) = c^n A$.

(i) False; let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then
det(A) = det(B) = 1, but det(A + B) = 0.

(j) True; let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so that $A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$ and

$$\begin{aligned}\det(A^2) &= (a^2 + bc)(bc + d^2) - (ab + bd)(ac + cd) \\ &= a^2bc + a^2d^2 + bcd^2 + b^2c^2 - (a^2bc + 2abcd + bcd^2) \\ &= a^2d^2 - 2abcd + b^2c^2 \\ &= (ad - bc)^2 \\ &= (\det(A))^2\end{aligned}$$

Section 2.2

Exercise Set 2.2

1. det(A) = (−2)(4) − (3)(1) = −8 − 3 = −11

$$A^T = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}$$

$$\det(A^T) = (-2)(4) - (1)(3) = -8 - 3 = -11$$

3.
$$\begin{aligned}\det(A) &= [(2)(2)(6) + (-1)(4)(5) + (3)(1)(-3)] - [(3)(2)(5) + (2)(4)(-3) + (-1)(1)(6)] \\ &= [24 - 20 - 9] - [30 - 24 - 6] \\ &= -5\end{aligned}$$

$$\begin{aligned}\det(A^T) &= [(2)(2)(6) + (1)(-3)(3) + (5)(-1)(4)] - [(5)(2)(3) + (2)(-3)(4) + (1)(-1)(6)] \\ &= [24 - 9 - 20] - [30 - 24 - 6] \\ &= -5\end{aligned}$$

5. The matrix is I_4 with the third row multiplied by −5.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -5$$

7. The matrix is I_4 with the second and third rows interchanged.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1$$

9. The matrix is I_4 with −9 times the fourth row added to the second row.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

11. $\begin{vmatrix} 0 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix}$

$= -\begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & -1 & -2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 3 & 1 \end{vmatrix}$

$= -\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 3 & 1 \end{vmatrix}$

$= -\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{vmatrix}$

$= -(-5)$

$= 5$

13. $\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix}$

$= 3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix}$

$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 3 & 4 \end{vmatrix}$

$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -11 \end{vmatrix}$

$= -3(-11)$

$= 33$

15. $\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$

$= -\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$

$= -\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 4 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 6 \end{vmatrix}$

$= 6$

17. $\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$

$= -\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$

$= -\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$

$= -\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$

$= -2$

19. Exercise 14: First add multiples of the first row to the remaining rows, then expand by cofactors along the first column. Repeat these steps with the 3×3 determinant, then evaluate the 2×2 determinant.

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & -9 & -2 \\ 0 & -3 & -1 \\ 12 & 0 & -1 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & -9 & -2 \\ 0 & -3 & -1 \\ 0 & 108 & 23 \end{vmatrix}$$
$$= \begin{vmatrix} -3 & -1 \\ 108 & 23 \end{vmatrix}$$
$$= (-3)(23) - (-1)(108)$$
$$= 39$$

Exercise 15: Add −2 times the second row to the first row, then expand by cofactors along the first column. For the 3×3 determinant, add multiples of the first row to the remaining row, then expand by cofactors of the first column. Finally, evaluate the 2×2 determinant.

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$
$$= (-1)\begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix}$$
$$= -\begin{vmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 1 & 4 \end{vmatrix}$$
$$= -\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix}$$
$$= -(-4-2)$$
$$= 6$$

Exercise 16: Add $-\frac{1}{2}$ times the first row to the second row, then expand by cofactors along the fourth column. For the 3×3 determinant, add −2 times the second row to the third row, then expand by cofactors along the second column. Finally, evaluate the 2×2 determinant.

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{vmatrix}$$
$$= (-1)\begin{vmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & 0 \end{vmatrix}$$
$$= -\begin{vmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{5}{3} & 0 & -\frac{2}{3} \end{vmatrix}$$
$$= -\left(\frac{1}{3}\right)\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{5}{3} & -\frac{2}{3} \end{vmatrix}$$
$$= -\frac{1}{3}\left[-\frac{1}{3} - \left(-\frac{5}{6}\right)\right]$$
$$= -\frac{1}{6}$$

Exercise 17: Add 2 times the first row to the second row, then expand by cofactors along the first column two times. For the 3×3 determinant, add −2 times the first row to the second row, then expand by cofactors along the first column. Finally, evaluate the 2×2 determinant.

$$\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} -1 & 2 & 6 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$
$$= (-1)\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$
$$= -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$
$$= -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$
$$= -[1-(-1)]$$
$$= -2$$

21. Interchange the second and third rows, then the first and second rows to arrive at the determinant whose value is given.

$$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

23. Remove the common factors of 3 from the first row, −1 from the second row, and 4 from the third row to arrive at the determinant whose value is given.

$$\begin{aligned} \begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} &= 3\begin{vmatrix} a & b & c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} \\ &= -3\begin{vmatrix} a & b & c \\ d & e & f \\ 4g & 4h & 4i \end{vmatrix} \\ &= -12\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \\ &= -12(-6) \\ &= 72 \end{aligned}$$

25. Add −1 times the third row to the first row to arrive at the matrix whose value is given.

$$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

27. Remove the common factor of −3 from the first row, then add 4 times the second row to the third to arrive at the matrix whose value is given.

$$\begin{aligned} &\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix} \\ &= -3\begin{vmatrix} a & b & c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix} \\ &= -3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \\ &= -3(-6) \\ &= 18 \end{aligned}$$

29.
$$\begin{aligned} &\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} \\ &= \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix} \\ &= (b-a)(c^2-a^2)-(c-a)(b^2-a^2) \\ &= (b-a)(c+a)(c-a)-(c-a)(b+a)(b-a) \\ &= (b-a)(c-a)[c+a-(b+a)] \\ &= (b-a)(c-a)(c-b) \end{aligned}$$

True/False 2.2

(a) True; interchanging two rows of a matrix causes the determinant to be multiplied by −1, so two such interchanges result in a multiplication by (−1)(−1) = 1.

(b) True; consider the transposes of the matrices so that the column operations performed on A to get B are row operations on A^T.

(c) False; adding a multiple of one row to another does not change the determinant. In this case, det(B) = det(A).

(d) False; multiplying each row by its row number causes the determinant to be multiplied by the row number, so det(B) = n! det(A).

(e) True; since the columns are identical, they are proportional, so det(A) = 0 by Theorem 2.2.5.

(f) True; −1 times the second and fourth rows can be added to the sixth row without changing det(A). Since this introduces a row of zeros, det(A) = 0.

Section 2.3

Exercise Set 2.3

1. det(A) = −4 − 6 = −10

$$\det(2A) = \begin{vmatrix} -2 & 4 \\ 6 & 8 \end{vmatrix} = -16-24 = -40 = 2^2(-10)$$

3. $\det(A) = [20 - 1 + 36] - [6 + 8 - 15] = 56$

$$\det(-2A) = \begin{vmatrix} -4 & 2 & -6 \\ -6 & -4 & -2 \\ -2 & -8 & -10 \end{vmatrix} = [-160+8-288]-[-48-64+120] = -448 = (-2)^3(56)$$

5. $$\det(AB) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix} = [18-170-0]-[80+0-62] = -170$$

$$\det(BA) = \begin{vmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{vmatrix} = [-22-120+510]-[660-20-102] = -170$$

$$\det(A) = 2\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 2(8-3) = 10$$

$$\det(B) = [1-10+0]-[15+0-7] = -17$$

$$\det(A+B) = \begin{vmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{vmatrix} = 5\begin{vmatrix} 3 & 3 \\ 5 & 3 \end{vmatrix} = 5(9-15) = -30 \neq \det(A)+\det(B)$$

7. Add multiples of the second row to the first and third row to simplify evaluating the determinant.

$$\det(A) = \begin{vmatrix} 0 & 3 & 5 \\ -1 & -1 & 0 \\ 0 & 2 & 3 \end{vmatrix} = -(-1)\begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} = 9-10 = -1$$

Since $\det(A) \neq 0$, A is invertible.

9. Expand by cofactors of the first column.

$$\det(A) = 2\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = 2(2) = 4$$

Since $\det(A) \neq 0$, A is invertible.

11. The third column of A is twice the first column, so $\det(A) = 0$ and A is not invertible.

13. Expand by cofactors of the first row.

$$\det(A) = 2\begin{vmatrix} 1 & 0 \\ 3 & 6 \end{vmatrix} = 2(6) = 12$$

Since $\det(A) \neq 0$, A is invertible.

15. $$\det(A) = (k-3)(k-2)-4 = k^2-5k+6-4 = k^2-5k+2$$

Use the quadratic formula to solve $k^2-5k+2=0$.

$$k = \frac{-(-5)\pm\sqrt{(-5)^2-4(1)(2)}}{2(1)} = \frac{5\pm\sqrt{17}}{2}$$

A is invertible for $k \neq \frac{5\pm\sqrt{17}}{2}$.

17. $$\det(A) = [2+12k+36]-[4k+18+12] = 8k+8$$

A is invertible for $k \neq -1$.

19. A is the same matrix as in Exercise 7, which was shown to be invertible ($\det(A) = -1$). The cofactors of A are $C_{11} = -3$, $C_{12} = -(-3) = 3$, $C_{13} = -4+2 = -2$, $C_{21} = -(15-20) = 5$, $C_{22} = 6-10 = -4$, $C_{23} = -(8-10) = 2$, $C_{31} = 0+5 = 5$, $C_{32} = -(0+5) = -5$, and $C_{33} = -2+5 = 3$.

The matrix of cofactors is $\begin{bmatrix} -3 & 3 & -2 \\ 5 & -4 & 2 \\ 5 & -5 & 3 \end{bmatrix}$.

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{-1}\begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{bmatrix}$$

21. A is the same matrix as in Exercise 9, which was shown to be invertible ($\det(A) = 4$). The cofactors of A are $C_{11} = 2$, $C_{12} = 0$, $C_{13} = 0$, $C_{21} = -(-6) = 6$, $C_{22} = 4$, $C_{23} = 0$, $C_{31} = 9-5 = 4$, $C_{32} = -(-6) = 6$, and $C_{33} = 2$.

The matrix of cofactors is $\begin{bmatrix} 2 & 0 & 0 \\ 6 & 4 & 0 \\ 4 & 6 & 2 \end{bmatrix}$.

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{4}\begin{bmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

23. Use row operations and cofactor expansion to simplify the determinant.

$$\begin{aligned}\det(A) &= \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 0 & 0 \\ 0 & 7 & 8 \\ 0 & 1 & 1 \end{vmatrix} \\ &= (-1)\begin{vmatrix} 7 & 8 \\ 1 & 1 \end{vmatrix} \\ &= -(7-8) \\ &= 1\end{aligned}$$

Since $\det(A) \neq 0$, A is invertible. The cofactors of A are

$$C_{11} = \begin{vmatrix} 5 & 2 & 2 \\ 3 & 8 & 9 \\ 3 & 2 & 2 \end{vmatrix} = -4$$

$$C_{12} = (-1)\begin{vmatrix} 2 & 2 & 2 \\ 1 & 8 & 9 \\ 1 & 2 & 2 \end{vmatrix} = -(-2) = 2$$

$$C_{13} = \begin{vmatrix} 2 & 5 & 2 \\ 1 & 3 & 9 \\ 1 & 3 & 2 \end{vmatrix} = -7$$

$$C_{14} = (-1)\begin{vmatrix} 2 & 5 & 2 \\ 1 & 3 & 8 \\ 1 & 3 & 2 \end{vmatrix} = -(-6) = 6$$

$$C_{21} = (-1)\begin{vmatrix} 3 & 1 & 1 \\ 3 & 8 & 9 \\ 3 & 2 & 2 \end{vmatrix} = -(-3) = 3$$

$$C_{22} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 8 & 9 \\ 1 & 2 & 2 \end{vmatrix} = -1$$

$$C_{23} = (-1)\begin{vmatrix} 1 & 3 & 1 \\ 1 & 3 & 9 \\ 1 & 3 & 2 \end{vmatrix} = 0$$

$$C_{24} = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 3 & 8 \\ 1 & 3 & 2 \end{vmatrix} = 0$$

$$C_{31} = \begin{vmatrix} 3 & 1 & 1 \\ 5 & 2 & 2 \\ 3 & 2 & 2 \end{vmatrix} = 0$$

$$C_{32} = (-1)\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{vmatrix} = 0$$

$$C_{33} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix} = -1$$

$$C_{34} = (-1)\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix} = -(-1) = 1$$

$$C_{41} = (-1)\begin{vmatrix} 3 & 1 & 1 \\ 5 & 2 & 2 \\ 3 & 8 & 9 \end{vmatrix} = -(1) = -1$$

$$C_{42} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 8 & 9 \end{vmatrix} = 0$$

$$C_{43} = (-1)\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 9 \end{vmatrix} = -(-8) = 8$$

$$C_{44} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 8 \end{vmatrix} = -7$$

The matrix of cofactors is $\begin{bmatrix} -4 & 2 & -7 & 6 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 8 & -7 \end{bmatrix}$.

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{1}\begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix} = \begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix}$$

25. $A = \begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$

$$\begin{aligned} \det(A) &= -2\begin{vmatrix} 4 & 5 \\ 1 & 5 \end{vmatrix} + 2\begin{vmatrix} 4 & 5 \\ 11 & 1 \end{vmatrix} \\ &= -2(20-5) + 2(4-55) \\ &= -132 \end{aligned}$$

$A_x = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$

$$\begin{aligned} \det(A_x) &= -2\begin{vmatrix} 2 & 5 \\ 1 & 5 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} \\ &= -2(10-5) + 2(2-15) \\ &= -36 \end{aligned}$$

$A_y = \begin{bmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

$$\begin{aligned} \det(A_y) &= -2\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 4 & 2 \\ 11 & 3 \end{vmatrix} \\ &= -2(4-2) + 2(12-22) \\ &= -24 \end{aligned}$$

$A_z = \begin{bmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$

$$\begin{aligned} \det(A_z) &= 4\begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} - 11\begin{vmatrix} 5 & 2 \\ 5 & 1 \end{vmatrix} + \begin{vmatrix} 5 & 2 \\ 1 & 3 \end{vmatrix} \\ &= 4(1-15) - 11(5-10) + (15-2) \\ &= 12 \end{aligned}$$

$$x = \frac{\det(A_x)}{\det(A)} = \frac{-36}{-132} = \frac{3}{11}$$

$$y = \frac{\det(A_y)}{\det(A)} = \frac{-24}{-132} = \frac{2}{11}$$

$$z = \frac{\det(A_z)}{\det(A)} = \frac{12}{-132} = -\frac{1}{11}$$

The solution is $x = \frac{3}{11}$, $y = \frac{2}{11}$, $z = -\frac{1}{11}$.

27. $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -1 & 0 \\ 4 & 0 & -3 \end{bmatrix}$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} + (-3)\begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} \\ &= (0+4) - 3(-1+6) \\ &= -11 \end{aligned}$$

$A_1 = \begin{bmatrix} 4 & -3 & 1 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

$$\det(A_1) = (-3)\begin{vmatrix} 4 & -3 \\ -2 & -1 \end{vmatrix} = -3(-4-6) = 30$$

$A_2 = \begin{bmatrix} 1 & 4 & 1 \\ 2 & -2 & 0 \\ 4 & 0 & -3 \end{bmatrix}$

$$\begin{aligned} \det(A_2) &= \begin{vmatrix} 2 & -2 \\ 4 & 0 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 4 \\ 2 & -2 \end{vmatrix} \\ &= (0+8) - 3(-2-8) \\ &= 38 \end{aligned}$$

$A_3 = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -1 & -2 \\ 4 & 0 & 0 \end{bmatrix}$

$$\det(A_3) = 4\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix} = 4(6+4) = 40$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{30}{-11} = -\frac{30}{11}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{38}{-11} = -\frac{38}{11}$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{40}{-11} = -\frac{40}{11}$$

The solution is $x_1 = -\frac{30}{11}$, $x_2 = -\frac{38}{11}$, $x_3 = -\frac{40}{11}$.

29. $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 7 & -2 \\ 2 & 6 & -1 \end{bmatrix}$

Note that the third row of A is the sum of the first and second rows. Thus, $\det(A) = 0$ and Cramer's rule does not apply.

31. $A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & 8 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

Use row operations and cofactor expansion to simplify the determinant.

$$\det(A) = \begin{vmatrix} 0 & -3 & -3 & -7 \\ 0 & 4 & -4 & -5 \\ 0 & -4 & -12 & -6 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$
$$= -(1)\begin{vmatrix} -3 & -3 & -7 \\ 4 & -4 & -5 \\ -4 & -12 & -6 \end{vmatrix}$$
$$= -\begin{vmatrix} -3 & -3 & -7 \\ 4 & -4 & -5 \\ 0 & -16 & -11 \end{vmatrix}$$
$$= -\left(-3\begin{vmatrix} -4 & -5 \\ -16 & -11 \end{vmatrix} - 4\begin{vmatrix} -3 & -7 \\ -16 & -11 \end{vmatrix}\right)$$
$$= 3(44-80)+4(33-112)$$
$$= -424$$
$$A_y = \begin{bmatrix} 4 & 6 & 1 & 1 \\ 3 & 1 & -1 & 1 \\ 7 & -3 & -5 & 8 \\ 1 & 3 & 1 & 2 \end{bmatrix}$$
$$\det(A_y) = \begin{vmatrix} 0 & -6 & -3 & 7 \\ 0 & -8 & -4 & -5 \\ 0 & -24 & -12 & -6 \\ 1 & 3 & 1 & 2 \end{vmatrix}$$
$$= -(1)\begin{vmatrix} -6 & -3 & 7 \\ -8 & -4 & -5 \\ -24 & -12 & -6 \end{vmatrix}$$
$$= -\begin{vmatrix} -6 & -3 & 7 \\ -8 & -4 & -5 \\ 0 & 0 & -34 \end{vmatrix}$$
$$= -(-34)\begin{vmatrix} -6 & -3 \\ -8 & -4 \end{vmatrix}$$
$$= -(-34)(24-24)$$
$$= 0$$
$$y = \frac{\det(A_y)}{\det(A)} = \frac{0}{-424} = 0$$

33. If all the entries in A are integers, then all the cofactors of A are integers, so all the entries in adj(A) are integers. Since det(A) = 1 and $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \operatorname{adj}(A),$ then all the entries in A^{-1} are integers.

35. **(a)** $\det(3A) = 3^3\det(A) = 27(-7) = -189$

(b) $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{-7} = -\frac{1}{7}$

(c) $\det(2A^{-1}) = 2^3\det(A^{-1}) = 8\left(-\frac{1}{7}\right) = -\frac{8}{7}$

(d)
$$\det((2A)^{-1}) = \frac{1}{\det(2A)} = \frac{1}{2^3\det(A)} = \frac{1}{8(-7)} = -\frac{1}{56}$$

(e)
$$\det\begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix} = \det\begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} = -\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -\det(A) = 7$$

37. **(a)** $\det(3A) = 3^3\det(A) = 27(7) = 189$

(b) $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{7}$

(c) $\det(2A^{-1}) = 2^3\det(A^{-1}) = 8\left(\frac{1}{7}\right) = \frac{8}{7}$

(d)
$$\det((2A)^{-1}) = \frac{1}{\det(2A)} = \frac{1}{2^3\det(A)} = \frac{1}{8(7)} = \frac{1}{56}$$

True/False 2.3

(a) False; $\det(2A) = 2^3\det(A)$ if A is a 3×3 matrix.

(b) False; consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Then det($A$) = det($B$) = 1, but det($A$ + B) = 0 ≠ det(2A) = 4.

(c) True; since A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)} \text{ and}$$

$$\det(A^{-1}BA) = \frac{1}{\det(A)} \cdot \det(B) \cdot \det(A) = \det(B).$$

(d) False; a square matrix A is invertible if and only if $\det(A) \neq 0$.

(e) True; since adj(A) is the transpose of the matrix of cofactors of A, and $(B^T)^T = B$ for any matrix B.

(f) True; the ij entry of $A \cdot \text{adj}(A)$ is $a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$. If $i \neq j$ then this sum is 0, and when $i = j$ then this sum is det(A). Thus, $A \cdot \text{adj}(A)$ is a diagonal matrix with diagonal entries all equal to det(A) or $(\det(A))I_n$.

(g) True; if det(A) were not 0, the system $A\mathbf{x} = \mathbf{0}$ would have exactly one solution (the trivial solution).

(h) True; if the reduced row echelon form of A were I_n, then there could be no such matrix $\mathbf{b}$.

(i) True; since E is invertible.

(j) True; if A is invertible, then so is A^{-1} and so both $A\mathbf{x} = \mathbf{0}$ and $A^{-1}\mathbf{x} = \mathbf{0}$ have only the trivial solution.

(k) True; if A is invertible, then $\det(A) \neq 0$ and A^{-1} is invertible, so $\det(A) \cdot A^{-1} = \text{adj}(A)$ is invertible.

(l) False; if the kth row of A is a row of zeros, then every cofactor $C_{ij} = 0$ for $i \neq k$, hence the cofactor matrix has a row of zeros and the corresponding column (not row) of adj(A) is a column of zeros.

Chapter 2 Supplementary Exercises

1. (a) $\begin{vmatrix} -4 & 2 \\ 3 & 3 \end{vmatrix} = -4(3) - 2(3) = -18$

(b)
$$\begin{aligned}
\begin{vmatrix} -4 & 2 \\ 3 & 3 \end{vmatrix} &= -\begin{vmatrix} 3 & 3 \\ -4 & 2 \end{vmatrix} \\
&= -3\begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix} \\
&= -3\begin{vmatrix} 1 & 1 \\ 0 & 6 \end{vmatrix} \\
&= -3(1)(6) \\
&= -18
\end{aligned}$$

3. (a)
$$\begin{aligned}
\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} &= -1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 5 & 2 \\ 2 & -1 \end{vmatrix} \\
&= -(2+1) - 3(-5-4) \\
&= 24
\end{aligned}$$

(b)
$$\begin{aligned}
\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} &= \begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ 0 & -14 & -5 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & -12 \end{vmatrix} \\
&= (-1)(2)(-12) \\
&= 24
\end{aligned}$$

5. (a)
$$\begin{aligned}
\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} &= 3\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ 4 & 2 \end{vmatrix} \\
&= 3(2-4) - (0+4) \\
&= -10
\end{aligned}$$

(b)
$$\begin{aligned}
\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} &= -\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & -1 \\ 0 & 4 & 2 \end{vmatrix} \\
&= -\begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 4 & 2 \end{vmatrix} \\
&= -\begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & -\frac{10}{3} \end{vmatrix} \\
&= -(1)(-3)\left(-\frac{10}{3}\right) \\
&= -10
\end{aligned}$$

7. (a)
$$\begin{vmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix} = 3\begin{vmatrix} 3 & 1 & 4 \\ 0 & -1 & 1 \\ 2 & -2 & 2 \end{vmatrix} - 6\begin{vmatrix} -2 & 1 & 4 \\ 1 & -1 & 1 \\ -9 & -2 & 2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 & 1 \\ 1 & 0 & -1 \\ -9 & 2 & -2 \end{vmatrix}$$
$$= 3\left(3\begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 4 \\ -1 & 1 \end{vmatrix}\right) - 6\left(-2\begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ -9 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & -1 \\ -9 & -2 \end{vmatrix}\right) - \left(-1\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} - (-1)\begin{vmatrix} -2 & 3 \\ -9 & 2 \end{vmatrix}\right)$$
$$= 3[3(-2+2) + 2(1+4)] - 6[-2(-2+2) - (2+9) + 4(-2-9)] - [-(-6-2) + (-4+27)]$$
$$= 30 + 330 - 31$$
$$= 329$$

(b)
$$\begin{vmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & -1 & 1 \\ -2 & 3 & 1 & 4 \\ 3 & 6 & 0 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix}$$
$$= -\begin{vmatrix} 1 & 0 & -1 & 1 \\ 0 & 3 & -1 & 6 \\ 0 & 6 & 3 & -2 \\ 0 & 2 & -11 & 11 \end{vmatrix}$$
$$= -\begin{vmatrix} 1 & 0 & -1 & 1 \\ 0 & 3 & -1 & 6 \\ 0 & 0 & 5 & -14 \\ 0 & 0 & -\frac{31}{3} & 7 \end{vmatrix}$$
$$= -\begin{vmatrix} 1 & 0 & -1 & 1 \\ 0 & 3 & -1 & 6 \\ 0 & 0 & 5 & -14 \\ 0 & 0 & 0 & -\frac{329}{15} \end{vmatrix}$$
$$= -(1)(3)(5)\left(-\frac{329}{15}\right)$$
$$= 329$$

9. Exercise 3:
$$\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix}\begin{matrix} -1 & 5 \\ 0 & 2 \\ -3 & 1 \end{matrix}$$
$$= [(-1)(2)(1) + (5)(-1)(-3) + (2)(0)(1)] - [(2)(2)(-3) + (-1)(-1)(1) + (5)(0)(1)]$$
$$= 13 - (-11)$$
$$= 24$$

Exercise 4:
$$\begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix} = \begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix}\begin{matrix} -1 & -2 \\ -4 & -5 \\ -7 & -8 \end{matrix}$$
$$= [(-1)(-5)(-9) + (-2)(-6)(-7) + (-3)(-4)(-8)] - [(-3)(-5)(-7) + (-1)(-6)(-8) + (-2)(-4)(-9)]$$
$$= -225 - (-225)$$
$$= 0$$

Exercise 5:

$$\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = \left|\begin{matrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{matrix}\right|\begin{matrix} 3 & 0 \\ 1 & 1 \\ 0 & 4 \end{matrix}$$
$$\begin{aligned} &= [(3)(1)(2)+(0)(1)(0)+(-1)(1)(4)]-[(-1)(1)(0)+(3)(1)(4)+(0)(1)(2)] \\ &= 2-12 \\ &= -10 \end{aligned}$$

Exercise 6:

$$\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = \left|\begin{matrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{matrix}\right|\begin{matrix} -5 & 1 \\ 3 & 0 \\ 1 & -2 \end{matrix}$$
$$\begin{aligned} &= [(-5)(0)(2)+(1)(2)(1)+(4)(3)(-2)]-[(4)(0)(1)+(-5)(2)(-2)+(1)(3)(2)] \\ &= -22-26 \\ &= -48 \end{aligned}$$

11. The determinants for Exercises 1, 3, and 4 were evaluated in Exercises 1, 3, and 9.
Exercise 2:

$$\begin{vmatrix} 7 & -1 \\ -2 & -6 \end{vmatrix} = 7(-6)-(-1)(-2) = -44$$

The determinants of the matrices in Exercises 1–3 are nonzero, so those matrices are invertible. The matrix in Exercise 4 is not invertible, since its determinant is zero.

13. $$\begin{aligned} \begin{vmatrix} 5 & b-3 \\ b-2 & -3 \end{vmatrix} &= 5(-3)-(b-3)(b-2) \\ &= -15-b^2+5b-6 \\ &= -b^2+5b-21 \end{aligned}$$

15. $$\begin{aligned} \begin{vmatrix} 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{vmatrix} &= (5)(2)(-1)(-4)(-3) \\ &= -120 \end{aligned}$$

17. Let $A = \begin{bmatrix} -4 & 2 \\ 3 & 3 \end{bmatrix}$. Then $\det(A) = -18$. The cofactors of A are $C_{11} = 3$, $C_{12} = -3$, $C_{21} = -2$, and $C_{22} = -4$. The matrix of cofactors is $\begin{bmatrix} 3 & -3 \\ -2 & -4 \end{bmatrix}$.

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{-18}\begin{bmatrix} 3 & -2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{9} \\ \frac{1}{6} & \frac{2}{9} \end{bmatrix}$$

19. Let $A = \begin{bmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{bmatrix}$. Then $\det(A) = 24$.

The cofactors of A are $C_{11} = 2+1 = 3$, $C_{12} = -(0-3) = 3$, $C_{13} = 0+6 = 6$, $C_{21} = -(5-2) = -3$, $C_{22} = -1+6 = 5$, $C_{23} = -(-1+15) = -14$, $C_{31} = -5-4 = -9$, $C_{32} = -(1-0) = -1$, and $C_{33} = -2-0 = -2$.

The matrix of cofactors is $\begin{bmatrix} 3 & 3 & 6 \\ -3 & 5 & -14 \\ -9 & -1 & -2 \end{bmatrix}$.

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{24}\begin{bmatrix} 3 & -3 & -9 \\ 3 & 5 & -1 \\ 6 & -14 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{8} & -\frac{3}{8} \\ \frac{1}{8} & \frac{5}{24} & -\frac{1}{24} \\ \frac{1}{4} & -\frac{7}{12} & -\frac{1}{12} \end{bmatrix}$$

23. Let $A = \begin{bmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{bmatrix}$.

Then det(A) = 329. The cofactors of A are $C_{11} = 10,\ C_{12} = 55,\ C_{13} = -21,\ C_{14} = -31,$ $C_{21} = -2,\ C_{22} = -11,\ C_{23} = 70,\ C_{24} = 72,$ $C_{31} = 52,\ C_{32} = -43,\ C_{33} = -175,\ C_{34} = 102,$ $C_{41} = -27,\ C_{42} = 16,\ C_{43} = -42,$ and $C_{44} = -15.$

The matrix of cofactors is

$$\begin{bmatrix} 10 & 55 & -21 & -31 \\ -2 & -11 & 70 & 72 \\ 52 & -43 & -175 & 102 \\ -27 & 16 & -42 & -15 \end{bmatrix}.$$

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{329}\begin{bmatrix} 10 & -2 & 52 & -27 \\ 55 & -11 & -43 & 16 \\ -21 & 70 & -175 & -42 \\ -31 & 72 & 102 & -15 \end{bmatrix} = \begin{bmatrix} \frac{10}{329} & -\frac{2}{329} & \frac{52}{329} & -\frac{27}{329} \\ \frac{55}{329} & -\frac{11}{329} & -\frac{43}{329} & \frac{16}{329} \\ -\frac{3}{47} & \frac{10}{47} & -\frac{25}{47} & -\frac{6}{47} \\ -\frac{31}{329} & \frac{72}{329} & \frac{102}{329} & -\frac{15}{329} \end{bmatrix}$$

25. In matrix form, the system is

$$\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$A = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

$$\det(A) = \frac{9}{25} + \frac{16}{25} = 1$$

$$A_{x'} = \begin{bmatrix} x & -\frac{4}{5} \\ y & \frac{3}{5} \end{bmatrix}$$

$$\det(A_{x'}) = \frac{3}{5}x + \frac{4}{5}y$$

$$A_{y'} = \begin{bmatrix} \frac{3}{5} & x \\ \frac{4}{5} & y \end{bmatrix}$$

$$\det(A_{y'}) = \frac{3}{5}y - \frac{4}{5}x = -\frac{4}{5}x + \frac{3}{5}y$$

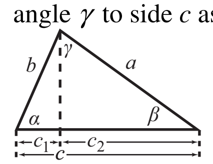
The solution is $x' = \frac{3}{5}x + \frac{4}{5}y,\quad y' = -\frac{4}{5}x + \frac{3}{5}y.$

27. $A = \begin{bmatrix} 1 & 1 & \alpha \\ 1 & 1 & \beta \\ \alpha & \beta & 1 \end{bmatrix}$

$$\det(A) = \begin{vmatrix} 1 & 1 & \alpha \\ 0 & 0 & \beta-\alpha \\ 0 & \beta-\alpha & 1-\alpha^2 \end{vmatrix} = \begin{vmatrix} 0 & \beta-\alpha \\ \beta-\alpha & 1-\alpha^2 \end{vmatrix} = 0 - (\beta-\alpha)^2 = -(\beta-\alpha)^2$$

det(A) = 0 if and only if $\alpha = \beta$, and the system has a nontrivial solution if and only if det(A) = 0.

29. **(a)** Draw the perpendicular from the vertex of angle γ to side c as shown.



Then $\cos\alpha = \frac{c_1}{b}$ and $\cos\beta = \frac{c_2}{a}$, so $c = c_1 + c_2 = a\cos\beta + b\cos\alpha$. A similar construction gives the other two equations in the system. The matrix form of the system is

$$\begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}\begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

$$\det(A)=\begin{vmatrix}0 & c & b\\ c & 0 & a\\ b & a & 0\end{vmatrix}$$
$$=-c\begin{vmatrix}c & a\\ b & 0\end{vmatrix}+b\begin{vmatrix}c & 0\\ b & a\end{vmatrix}$$
$$=-c(0-ab)+b(ac-0)$$
$$=2abc$$

$$A_\alpha=\begin{bmatrix}a & c & b\\ b & 0 & a\\ c & a & 0\end{bmatrix}$$

$$\det(A_\alpha)=-c\begin{vmatrix}b & a\\ c & 0\end{vmatrix}-a\begin{vmatrix}a & b\\ b & a\end{vmatrix}$$
$$=-c(0-ac)-a(a^2-b^2)$$
$$=a(c^2+b^2-a^2)$$

$$\cos\alpha=\frac{\det(A_\alpha)}{\det(A)}$$
$$=\frac{a(c^2+b^2-a^2)}{2abc}$$
$$=\frac{c^2+b^2-a^2}{2bc}$$

(b) $$A_\beta=\begin{bmatrix}0 & a & b\\ c & b & a\\ b & c & 0\end{bmatrix}$$

$$\det(A_\beta)=-a\begin{vmatrix}c & a\\ b & 0\end{vmatrix}+b\begin{vmatrix}c & b\\ b & c\end{vmatrix}$$
$$=-a(0-ab)+b(c^2-b^2)$$
$$=b(a^2+c^2-b^2)$$

$$\cos\beta=\frac{\det(A_\beta)}{\det(A)}$$
$$=\frac{b(a^2+c^2-b^2)}{2abc}$$
$$=\frac{a^2+c^2-b^2}{2ac}$$

$$A_\gamma=\begin{bmatrix}0 & c & a\\ c & 0 & b\\ b & a & c\end{bmatrix}$$

$$\det(A_\gamma)=-c\begin{vmatrix}c & b\\ b & c\end{vmatrix}+a\begin{vmatrix}c & 0\\ b & a\end{vmatrix}$$
$$=-c(c^2-b^2)+a(ac-0)$$
$$=c(a^2+b^2-c^2)$$

$$\cos\gamma=\frac{\det(A_\gamma)}{\det(A)}=\frac{c(a^2+b^2-c^2)}{2abc}$$
$$=\frac{a^2+b^2-c^2}{2ab}$$

31. If A is invertible then A^{-1} is also invertible and $\det(A)\neq 0$, so $\det(A)A^{-1}$ is invertible. Thus, $\operatorname{adj}(A)=\det(A)A^{-1}$ is invertible and

$$(\operatorname{adj}(A))^{-1}=(\det(A)A^{-1})^{-1}$$
$$=\frac{1}{\det(A)}(A^{-1})^{-1}$$
$$=\frac{1}{\det(A)}A$$
$$=\operatorname{adj}(\mathrm{A}^{-1})$$

since $\operatorname{adj}(A^{-1})=\det(A^{-1})(A^{-1})^{-1}=\dfrac{1}{\det(A)}A.$

33. Let A be an $n\times n$ matrix for which the sum of the entries in each row is zero and let $\mathbf{x}_1$ be the $n\times 1$ column matrix whose entries are all one.

Then $A\mathbf{x}_1=\begin{bmatrix}0\\ 0\\ \vdots\\ 0\end{bmatrix}$ because each entry in the product is the sum of the entries in a row of A. That is, $A\mathbf{x}=\mathbf{0}$ has a nontrivial solution, so $\det(A)=0$.

35. Add 10,000 times the first column, 1000 times the second column, 100 times the third column, and 10 times the fourth column to the fifth column. The entries in the fifth column are now 21,375, 38,798, 34,162, 40,223, and 79,154, respectively. Evaluating the determinant by expanding by cofactors of the fifth column, gives

$$21{,}375C_{15}+38{,}798C_{25}+34{,}162C_{35}$$
$$+40{,}223C_{45}+79{,}154C_{55}.$$

Since each coefficient of C_{i5} is divisible by 19, this sum must also be divisible by 19.
Note that the column operations do not change the value of the determinant—consider them as row operations on the transpose.

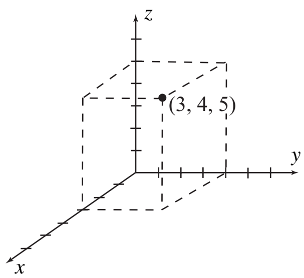
Chapter 3

Euclidean Vector Spaces

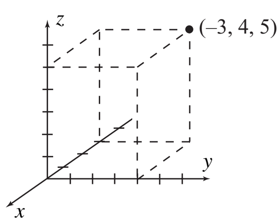
Section 3.1

Exercise Set 3.1

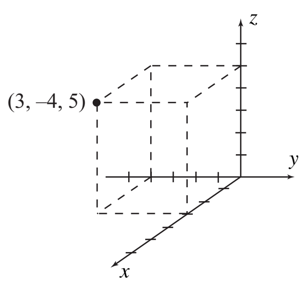
1. **(a)**



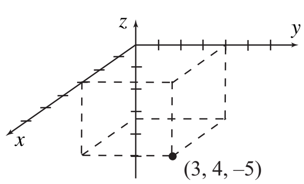
(b)



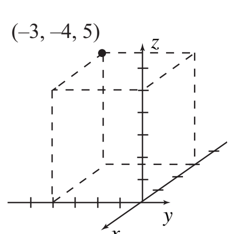
(c)



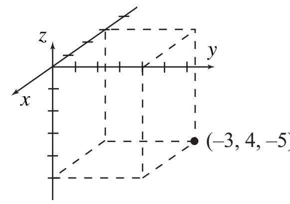
(d)



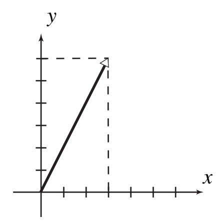
(e)



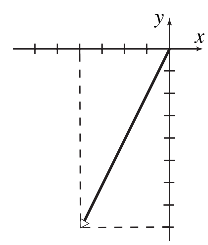
(f)



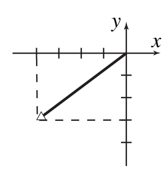
3. **(a)**



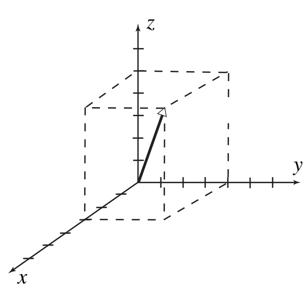
(b)



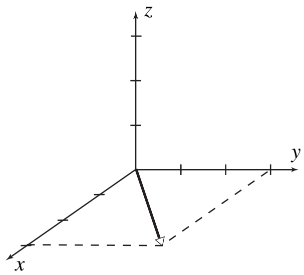
(c)



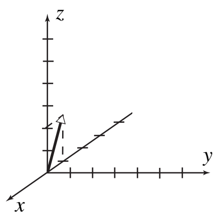
(d)



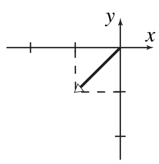
(e)



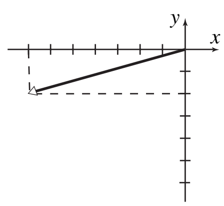
(f)



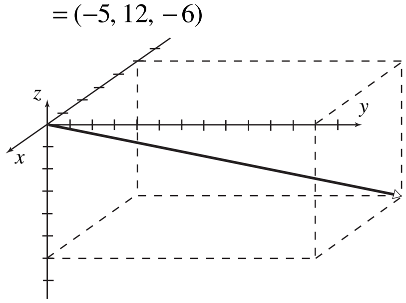
5. (a) $\overrightarrow{P_1P_2} = (3-4, 7-8) = (-1, -1)$



(b) $\overrightarrow{P_1P_2} = (-4-3, -7-(-5)) = (-7, -2)$



(c) $\overrightarrow{P_1P_2} = (-2-3, 5-(-7), -4-2)$
$= (-5, 12, -6)$



7. (a) $\overrightarrow{P_1P_2} = (2-3, 8-5) = (-1, 3)$

(b) $\overrightarrow{P_1P_2} = (2-5, 4-(-2), 2-1) = (-3, 6, 1)$

9. (a) Let the terminal point be $B(b_1, b_2)$. Then $(b_1-1, b_2-1) = (1, 2)$ so $b_1 = 2$ and $b_2 = 3$. The terminal point is $B(2, 3)$.

(b) Let the initial point be $A(a_1, a_2, a_3)$. Then $(-1-a_1, -1-a_2, 2-a_3) = (1, 1, 3)$, so $a_1 = -2$, $a_2 = -2$, and $a_3 = -1$. The initial point is $A(-2, -2, -1)$.

11. (a) One possibility is $\mathbf{u} = \mathbf{v}$, then the initial point (x, y, z) satisfies
$(3 - x, 0 - y, -5 - z) = (4, -2, -1)$, so that
$x = -1$, $y = 2$, and $z = -4$.
One possibility is $(-1, 2, -4)$.

(b) One possibility is $\mathbf{u} = -\mathbf{v}$, then the initial point (x, y, z) satisfies
$(3 - x, 0 - y, -5 - z) = (-4, 2, 1)$, so that
$x = 7$, $y = -2$, and $z = -6$.
One possibility is $(7, -2, -6)$.

13. (a) $\mathbf{u} + \mathbf{w} = (4, -1) + (-3, -3) = (1, -4)$

(b) $\mathbf{v} - 3\mathbf{u} = (0, 5) - (12, -3) = (-12, 8)$

(c)
$$\begin{aligned} 2(\mathbf{u}-5\mathbf{w}) &= 2((4, -1)-(-15, -15)) \\ &= 2(19, 14) \\ &= (38, 28) \end{aligned}$$

(d)
$$\begin{aligned} &3\mathbf{v}-2(\mathbf{u}+2\mathbf{w}) \\ &= (0, 15)-2((4, -1)+(-6, -6)) \\ &= (0, 15)-2(-2, -7) \\ &= (0, 15)+(4, 14) \\ &= (4, 29) \end{aligned}$$

(e)
$$\begin{aligned} &-3(\mathbf{w}-2\mathbf{u}+\mathbf{v}) \\ &= -3((-3, -3)-(8, -2)+(0, 5)) \\ &= -3(-11, 4) \\ &= (33, -12) \end{aligned}$$

(f)
$$\begin{aligned} &(-2\mathbf{u} - \mathbf{v}) - 5(\mathbf{v} + 3\mathbf{w}) \\ &= ((-8, 2) - (0, 5)) - 5((0, 5) + (-9, -9)) \\ &= (-8, -3) - 5(-9, -4) \\ &= (-8, -3) + (45, 20) \\ &= (37, 17) \end{aligned}$$

15. (a)
$$\begin{aligned} \mathbf{v}-\mathbf{w} &= (4, 7, -3, 2)-(5, -2, 8, 1) \\ &= (-1, 9, -11, 1) \end{aligned}$$

(b)
$$\begin{aligned} 2\mathbf{u}+7\mathbf{v} &= (-6, 4, 2, 0)+(28, 49, -21, 14) \\ &= (22, 53, -19, 14) \end{aligned}$$

(c) $-\mathbf{u}+(\mathbf{v}-4\mathbf{w}) = (3, -2, -1, 0) + ((4, 7, -3, 2) - (20, -8, 32, 4))$
$= (3, -2, -1, 0) + (-16, 15, -35, -2)$
$= (-13, 13, -36, -2)$

(d) $6(\mathbf{u}-3\mathbf{v}) = 6((-3, 2, 1, 0)-(12, 21, -9, 6))$
$= 6(-15, -19, 10, -6)$
$= (-90, -114, 60, -36)$

(e) $-\mathbf{v}-\mathbf{w} = (-4, -7, 3, -2)-(5, -2, 8, 1)$
$= (-9, -5, -5, -3)$

(f) $(6\mathbf{v}-\mathbf{w})-(4\mathbf{u}+\mathbf{v}) = ((24, 42, -18, 12) - (5, -2, 8, 1)) - ((-12, 8, 4, 0) + (4, 7, -3, 2))$
$= (19, 44, -26, 11) - (-8, 15, 1, 2)$
$= (27, 29, -27, 9)$

17. **(a)** $\mathbf{w}-\mathbf{u} = (-4, 2, -3, -5, 2)-(5, -1, 0, 3, -3)$
$= (-9, 3, -3, -8, 5)$

(b) $2\mathbf{v}+3\mathbf{u} = (-2, -2, 14, 4, 0)+(15, -3, 0, 9, -9)$
$= (13, -5, 14, 13, -9)$

(c) $-\mathbf{w}+3(\mathbf{v}-\mathbf{u}) = (4, -2, 3, 5, -2) + 3((-1, -1, 7, 2, 0) - (5, -1, 0, 3, -3))$
$= (4, -2, 3, 5, -2) + 3(-6, 0, 7, -1, 3)$
$= (4, -2, 3, 5, -2) + (-18, 0, 21, -3, 9)$
$= (-14, -2, 24, 2, 7)$

(d) $5(-\mathbf{v}+4\mathbf{u}-\mathbf{w}) = 5((1, 1, -7, -2, 0) + (20, -4, 0, 12, -12) - (-4, 2, -3, -5, 2))$
$= 5(25, -5, -4, 15, -14)$
$= (125, -25, -20, 75, -70)$

(e) $-2(3\mathbf{w}+\mathbf{v})+(2\mathbf{u}+\mathbf{w}) = -2((-12, 6, -9, -15, 6)+(-1, -1, 7, 2, 0))+((10, -2, 0, 6, -6)+(-4, 2, -3, -5, 2))$
$= -2(-13, 5, -2, -13, 6)+(6, 0, -3, 1, -4)$
$= (26, -10, 4, 26, -12)+(6, 0, -3, 1, -4)$
$= (32, -10, 1, 27, -16)$

(f) $\frac{1}{2}(\mathbf{w}-5\mathbf{v}+2\mathbf{u})+\mathbf{v} = \frac{1}{2}((-4, 2, -3, -5, 2)-(-5, -5, 35, 10, 0)+(10, -2, 0, 6, -6))+(-1, -1, 7, 2, 0)$
$= \frac{1}{2}(11, 5, -38, -9, -4)+(-1, -1, 7, 2, 0)$
$= \left(\frac{11}{2}, \frac{5}{2}, -19, -\frac{9}{2}, -2\right)+(-1, -1, 7, 2, 0)$
$= \left(\frac{9}{2}, \frac{3}{2}, -12, -\frac{5}{2}, -2\right)$

19. **(a)** $\mathbf{v}-\mathbf{w} = (4, 0, -8, 1, 2)-(6, -1, -4, 3, -5)$
$= (-2, 1, -4, -2, 7)$

(b) $6\mathbf{u}+2\mathbf{v} = (-18, 6, 12, 24, 24)+(8, 0, -16, 2, 4)$
$= (-10, 6, -4, 26, 28)$

(c) $(2\mathbf{u}-7\mathbf{w})-(8\mathbf{v}+\mathbf{u}) = ((-6, 2, 4, 8, 8)-(42, -7, -28, 21, -35))-((32, 0, -64, 8, 16)+(-3, 1, 2, 4, 4))$
$= (-48, 9, 32, -13, 43)-(29, 1, -62, 12, 20)$
$= (-77, 8, 94, -25, 23)$

21. Let $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$.

$$\begin{aligned}2\mathbf{u}-\mathbf{v}+\mathbf{x} &= (-6, 2, 4, 8, 8)-(4, 0, -8, 1, 2)+(x_1, x_2, x_3, x_4, x_5)\\ &= (-10+x_1, 2+x_2, 12+x_3, 7+x_4, 6+x_5)\end{aligned}$$

$$\begin{aligned}7\mathbf{x}+\mathbf{w} &= (7x_1, 7x_2, 7x_3, 7x_4, 7x_5)+(6, -1, -4, 3, -5)\\ &= (7x_1+6, 7x_2-1, 7x_3-4, 7x_4+3, 7x_5-5)\end{aligned}$$

Equate components.

$-10+x_1 = 7x_1+6 \Rightarrow x_1 = -\dfrac{8}{3}$

$2+x_2 = 7x_2-1 \Rightarrow x_2 = \dfrac{1}{2}$

$12+x_3 = 7x_3-4 \Rightarrow x_3 = \dfrac{8}{3}$

$7+x_4 = 7x_4+3 \Rightarrow x_4 = \dfrac{2}{3}$

$6+x_5 = 7x_5-5 \Rightarrow x_5 = \dfrac{11}{6}$

$\mathbf{x} = \left(-\dfrac{8}{3}, \dfrac{1}{2}, \dfrac{8}{3}, \dfrac{2}{3}, \dfrac{11}{6}\right)$

23. (a) There is no scalar k such that $k\mathbf{u}$ is the given vector, so the given vector is not parallel to $\mathbf{u}$.

(b) $-2\mathbf{u} = (4, -2, 0, -6, -10, -2)$
The given vector is parallel to $\mathbf{u}$.

(c) The given vector is $\mathbf{0}$, which is parallel to all vectors.

25.
$$\begin{aligned}a\mathbf{u}+b\mathbf{v} &= (a, -a, 3a, 5a)+(2b, b, 0, -3b)\\ &= (a+2b, -a+b, 3a, 5a-3b)\\ &= (1, -4, 9, 18)\end{aligned}$$

Equating the third components give $3a = 9$ or $a = 3$. Equating any other components gives $b = -1$. The scalars are $a = 3$, $b = -1$.

27. Equating components gives the following system of equations.

$$\begin{aligned}c_1+3c_2 &= -1\\ -c_1+2c_2+c_3 &= 1\\ c_2+4c_3 &= 19\end{aligned}$$

The reduced row echelon form of $\begin{bmatrix} 1 & 3 & 0 & -1\\ -1 & 2 & 1 & 1\\ 0 & 1 & 4 & 19\end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 2\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 5\end{bmatrix}$.

The scalars are $c_1 = 2$, $c_2 = -1$, $c_3 = 5$.

29. Equating components gives the following system of equations.

$$\begin{aligned} -c_1+2c_2+7c_3+6c_4&=0\\ 3c_1 \qquad\; +c_3+3c_4&=5\\ 2c_1+4c_2+c_3+c_4&=6\\ -c_2+4c_3+2c_4&=-3 \end{aligned}$$

The reduced row echelon form of

$$\begin{bmatrix} -1 & 2 & 7 & 6 & 0\\ 3 & 0 & 1 & 3 & 5\\ 2 & 4 & 1 & 1 & 6\\ 0 & -1 & 4 & 2 & -3 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The scalars are $c_1=1,\ c_2=1,\ c_3=-1,\ c_4=1.$

31. Equating components gives the following system of equations.

$$\begin{aligned} -2c_1-3c_2+c_3&=0\\ 9c_1+2c_2+7c_3&=5\\ 6c_1+c_2+5c_3&=4 \end{aligned}$$

The reduced row echelon form of

$$\begin{bmatrix} -2 & -3 & 1 & 0\\ 9 & 2 & 7 & 5\\ 6 & 1 & 5 & 4 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & 0 & 1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the bottom row, the system has no solution, so no such scalars exist.

33. (a)
$$\begin{aligned} \frac{1}{2}\overrightarrow{PQ} &= \frac{1}{2}(7-2,\ -4-3,\ 1-(-2))\\ &= \frac{1}{2}(5,\ -7,\ 3)\\ &= \left(\frac{5}{2},\ -\frac{7}{2},\ \frac{3}{2}\right) \end{aligned}$$

Locating $\frac{1}{2}\overrightarrow{PQ}$ with its initial point at P gives the midpoint.

$$\left(2+\frac{5}{2},\ 3+\left(-\frac{7}{2}\right),\ -2+\frac{3}{2}\right)=\left(\frac{9}{2},\ -\frac{1}{2},\ -\frac{1}{2}\right)$$

(b) $\frac{3}{4}\overrightarrow{PQ}=\frac{3}{4}(5,\ -7,\ 3)=\left(\frac{15}{4},\ -\frac{21}{4},\ \frac{9}{4}\right)$

Locating $\frac{3}{4}\overrightarrow{PQ}$ with its initial point at P gives the desired point.

$$\begin{aligned} &\left(2+\frac{15}{4},\ 3+\left(-\frac{21}{4}\right),\ -2+\frac{9}{4}\right)\\ &=\left(\frac{23}{4},\ -\frac{9}{4},\ \frac{1}{4}\right) \end{aligned}$$

True/False 3.1

(a) False; vector equivalence is determined solely by length and direction, not location.

(b) False; (a, b) is a vector in 2-space, while $(a, b, 0)$ is a vector in 3-space. Alternatively, equivalent vectors must have the same number of components.

(c) False; $\mathbf{v}$ and $k\mathbf{v}$ are parallel for all values of k.

(d) True; apply Theorem 3.1.1 parts (a), (b), and (a) again:
$$\begin{aligned} \mathbf{v}+(\mathbf{u}+\mathbf{w}) &= \mathbf{v}+(\mathbf{w}+\mathbf{u})\\ &=(\mathbf{v}+\mathbf{w})+\mathbf{u}\\ &=(\mathbf{w}+\mathbf{v})+\mathbf{u}. \end{aligned}$$

(e) True; the vector $-\mathbf{u}$ can be added to both sides of the equation.

(f) False; a and b must be nonzero scalars for the statement to be true.

(g) False; the vectors $\mathbf{v}$ and $-\mathbf{v}$ have the same length and can be placed to be collinear, but are not equal.

(h) True; vector addition is defined component-wise.

(i) False; $(k+m)(\mathbf{u}+\mathbf{v})=(k+m)\mathbf{u}+(k+m)\mathbf{v}$.

(j) True; $\mathbf{x}=\frac{1}{8}(5\mathbf{v}+4\mathbf{w})$

(k) False; for example, if $\mathbf{v}_2=-\mathbf{v}_1$ then $a_1\mathbf{v}_1+a_2\mathbf{v}_2=b_1\mathbf{v}_1+b_2\mathbf{v}_2$ as long as $a_1-a_2=b_1-b_2$.

Section 3.2

Exercise Set 3.2

1. (a) $\|\mathbf{v}\|=\sqrt{4^2+(-3)^2}=\sqrt{25}=5$

$\mathbf{u}_1=\frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{1}{5}(4,\ -3)=\left(\frac{4}{5},\ -\frac{3}{5}\right)$ has the same direction as $\mathbf{v}$.

$\mathbf{u}_2=-\frac{\mathbf{v}}{\|\mathbf{v}\|}=-\frac{1}{5}(4,\ -3)=\left(-\frac{4}{5},\ \frac{3}{5}\right)$ has the direction opposite $\mathbf{v}$.

(b) $\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$

$\mathbf{u}_1 = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{2\sqrt{3}}(2, 2, 2) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ has the same direction as **v**.

$$\begin{aligned}\mathbf{u}_2 &= -\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= -\frac{1}{2\sqrt{3}}(2, 2, 2) \\ &= \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\end{aligned}$$

has the direction opposite **v**.

(c) $\|\mathbf{v}\| = \sqrt{1^2 + 0^2 + 2^2 + 1^2 + 3^2} = \sqrt{15}$

$$\begin{aligned}\mathbf{u}_1 &= \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= \frac{1}{\sqrt{15}}(1, 0, 2, 1, 3) \\ &= \left(\frac{1}{\sqrt{15}}, 0, \frac{2}{\sqrt{15}}, \frac{1}{\sqrt{15}}, \frac{3}{\sqrt{15}}\right)\end{aligned}$$

has the same direction as **v**.

$$\begin{aligned}\mathbf{u}_2 &= -\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= -\frac{1}{\sqrt{15}}(1, 0, 2, 1, 3) \\ &= \left(-\frac{1}{\sqrt{15}}, 0, -\frac{2}{\sqrt{15}}, -\frac{1}{\sqrt{15}}, -\frac{3}{\sqrt{15}}\right)\end{aligned}$$

has the direction opposite **v**.

3. (a)
$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\| &= \|(2, -2, 3) + (1, -3, 4)\| \\ &= \|(3, -5, 7)\| \\ &= \sqrt{3^2 + (-5)^2 + 7^2} \\ &= \sqrt{83}\end{aligned}$$

(b)
$$\begin{aligned}&\|\mathbf{u}\| + \|\mathbf{v}\| \\ &= \|(2, -2, 3)\| + \|(1, -3, 4)\| \\ &= \sqrt{2^2 + (-2)^2 + 3^2} + \sqrt{1^2 + (-3)^2 + 4^2} \\ &= \sqrt{17} + \sqrt{26}\end{aligned}$$

(c)
$$\begin{aligned}\|-2\mathbf{u} + 2\mathbf{v}\| &= \|(-4, 4, -6) + (2, -6, 8)\| \\ &= \|(-2, -2, 2)\| \\ &= \sqrt{(-2)^2 + (-2)^2 + 2^2} \\ &= \sqrt{12} \\ &= 2\sqrt{3}\end{aligned}$$

(d) $\|3\mathbf{u}-5\mathbf{v}+\mathbf{w}\|$
$= \|(6, -6, 9)-(5, -15, 20)+(3, 6, -4)\|$
$= \|(4, 15, -15)\|$
$= \sqrt{4^2+15^2+(-15)^2}$
$= \sqrt{466}$

5. (a) $\|3\mathbf{u}-5\mathbf{v}+\mathbf{w}\| = \|(-6, -3, 12, 15)-(15, 5, -25, 35)+(-6, 2, 1, 1)\|$
$= \|(-27, -6, 38, -19)\|$
$= \sqrt{(-27)^2+(-6)^2+38^2+(-19)^2}$
$= \sqrt{2570}$

(b) $\|3\mathbf{u}\|-5\|\mathbf{v}\|+\|\mathbf{w}\| = \|(-6, -3, 12, 15)\|-5\|(3, 1, -5, 7)\|+\|(-6, 2, 1, 1)\|$
$= \sqrt{(-6)^2+(-3)^2+12^2+15^2}-5\sqrt{3^2+1^2+(-5)^2+7^2}+\sqrt{(-6)^2+2^2+1^2+1^2}$
$= \sqrt{414}-5\sqrt{84}+\sqrt{42}$
$= 3\sqrt{46}-10\sqrt{21}+\sqrt{42}$

(c) $\|-\|\mathbf{u}\|\mathbf{v}\| = \left\|-\sqrt{(-2)^2+(-1)^2+4^2+5^2}(3, 1, -5, 7)\right\|$
$= \|-\sqrt{46}(3, 1, -5, 7)\|$
$= \sqrt{46}\sqrt{3^2+1^2+(-5)^2+7^2}$
$= \sqrt{46}\sqrt{84}$
$= 2\sqrt{966}$

7. $\|k\mathbf{v}\| = |k|\|(-2, 3, 0, 6)\|$
$= |k|\sqrt{(-2)^2+3^2+0+6^2}$
$= |k|\sqrt{49}$
$= 7|k|$

$7|k| = 5$ if $|k| = \frac{5}{7}$, so $k = \frac{5}{7}$ or $k = -\frac{5}{7}$.

9. (a) $\mathbf{u}\cdot\mathbf{v} = (3)(2)+(1)(2)+(4)(-4) = -8$
$\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|^2 = 3^2+1^2+4^2 = 26$
$\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2 = 2^2+2^2+(-4)^2 = 24$

(b) $\mathbf{u}\cdot\mathbf{v} = (1)(2)+(1)(-2)+(4)(3)+(6)(-2)$
$= 0$
$\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|^2 = 1^2+1^2+4^2+6^2 = 54$
$\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2 = 2^2+(-2)^2+3^2+(-2)^2 = 21$

11. **(a)** $$\begin{aligned} d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\| \\ &= \sqrt{(3-1)^2+(3-0)^2+(3-4)^2} \\ &= \sqrt{2^2+3^2+(-1)^2} \\ &= \sqrt{14} \end{aligned}$$

(b) $$\begin{aligned} d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\| \\ &= \sqrt{(0-(-3))^2+(-2-2)^2+(-1-4)^2+(1-4)^2} \\ &= \sqrt{3^2+(-4)^2+(-5)^2+(-3)^2} \\ &= \sqrt{59} \end{aligned}$$

(c) $$\begin{aligned} d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\| \\ &= \sqrt{(3-(-4))^2+(-3-1)^2+(-2-(-1))^2+(0-5)^2+(-3-0)^2+(13-(-11))^2+(5-4)^2} \\ &= \sqrt{7^2+(-4)^2+(-1)^2+(-5)^2+(-3)^2+24^2+1^2} \\ &= \sqrt{677} \end{aligned}$$

13. **(a)** $$\begin{aligned} \cos\theta &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \\ &= \frac{(3)(1)+(3)(0)+(3)(4)}{\sqrt{3^2+3^2+3^2}\sqrt{1^2+0^2+4^2}} \\ &= \frac{15}{\sqrt{27}\sqrt{17}} \end{aligned}$$

Since cos θ is positive, θ is acute.

(b) $$\begin{aligned} \cos\theta &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \\ &= \frac{(0)(-3)+(-2)(2)+(-1)(4)+(1)(4)}{\sqrt{0^2+(-2)^2+(-1)^2+1^2}\sqrt{(-3)^2+2^2+4^2+4^2}} \\ &= \frac{-4}{\sqrt{6}\sqrt{45}} \\ &= -\frac{4}{\sqrt{6}\sqrt{45}} \end{aligned}$$

Since cos θ is negative, θ is obtuse.

(c) $$\begin{aligned} \cos\theta &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \\ &= \frac{(3)(-4)+(-3)(1)+(-2)(-1)+(0)(5)+(-3)(0)+(13)(-11)+(5)(4)}{\sqrt{3^2+(-3)^2+(-2)^2+0^2+(-3)^2+13^2+5^2}\sqrt{(-4)^2+1^2+(-1)^2+5^2+0^2+(-11)^2+4^2}} \\ &= \frac{-136}{\sqrt{225}\sqrt{180}} \\ &= -\frac{136}{\sqrt{225}\sqrt{180}} \end{aligned}$$

Since cos θ is negative, θ is obtuse.

15. The angle θ between the vectors is 30°.

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta = 9\cdot 5\cos 30° = 45\frac{\sqrt{3}}{2}$$

17. (a) $\mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w})$ does not make sense because $\mathbf{v}\cdot\mathbf{w}$ is a scalar.

(b) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})$ makes sense.

(c) $\|\mathbf{u}\cdot\mathbf{v}\|$ does not make sense because $\mathbf{u}\cdot\mathbf{v}$ is a scalar.

(d) $(\mathbf{u}\cdot\mathbf{v})-\|\mathbf{u}\|$ makes sense since both $\mathbf{u}\cdot\mathbf{v}$ and $\|\mathbf{u}\|$ are scalars.

19. (a)
$$\frac{(-4,-3)}{\|(-4,-3)\|} = \frac{(-4,-3)}{\sqrt{(-4)^2+(-3)^2}} = \frac{(-4,-3)}{\sqrt{25}} = \left(-\frac{4}{5}, -\frac{3}{5}\right)$$

(b)
$$\frac{(1,7)}{\|(1,7)\|} = \frac{(1,7)}{\sqrt{1^2+7^2}} = \frac{(1,7)}{\sqrt{50}} = \left(\frac{1}{5\sqrt{2}}, \frac{7}{5\sqrt{2}}\right)$$

(c)
$$\frac{\left(-3, 2, \sqrt{3}\right)}{\left\|\left(-3, 2, \sqrt{3}\right)\right\|} = \frac{\left(-3, 2, \sqrt{3}\right)}{\sqrt{(-3)^2+2^2+\left(\sqrt{3}\right)^2}} = \frac{\left(-3, 2, \sqrt{3}\right)}{\sqrt{16}} = \left(-\frac{3}{4}, \frac{1}{2}, \frac{\sqrt{3}}{4}\right)$$

(d)
$$\frac{(1, 2, 3, 4, 5)}{\|(1, 2, 3, 4, 5)\|} = \frac{(1, 2, 3, 4, 5)}{\sqrt{1^2+2^2+3^2+4^2+5^2}} = \frac{(1, 2, 3, 4, 5)}{\sqrt{55}} = \left(\frac{1}{\sqrt{55}}, \frac{2}{\sqrt{55}}, \frac{3}{\sqrt{55}}, \frac{4}{\sqrt{55}}, \frac{5}{\sqrt{55}}\right)$$

21. Divide $\mathbf{v}$ by $\|\mathbf{v}\|$ to get a unit vector that has the same direction as $\mathbf{v}$, then multiply by m: $m\frac{\mathbf{v}}{\|\mathbf{v}\|}$.

23. (a)
$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(2)(5)+(3)(-7)}{\sqrt{2^2+3^2}\sqrt{5^2+(-7)^2}} = \frac{-11}{\sqrt{13}\sqrt{74}} = -\frac{11}{\sqrt{962}}$$

(b)
$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(-6)(4)+(-2)(0)}{\sqrt{(-6)^2+(-2)^2}\sqrt{4^2+0^2}} = \frac{-24}{\sqrt{40}\sqrt{16}} = -\frac{24}{8\sqrt{10}} = -\frac{3}{\sqrt{10}}$$

(c)
$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(1)(3)+(-5)(3)+(4)(3)}{\sqrt{1^2+(-5)^2+4^2}\sqrt{3^2+3^2+3^2}} = \frac{0}{\sqrt{42}\sqrt{27}} = 0$$

(d)
$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(-2)(1)+(2)(7)+(3)(-4)}{\sqrt{(-2)^2+2^2+3^2}\sqrt{1^2+7^2+(-4)^2}} = \frac{0}{\sqrt{17}\sqrt{66}} = 0$$

25. (a) $|\mathbf{u}\cdot\mathbf{v}| = |(3)(4)+(2)(-1)| = 10$

$$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{3^2+2^2}\sqrt{4^2+(-1)^2} = \sqrt{13}\sqrt{17} \approx 14.866$$

(b) $|\mathbf{u}\cdot\mathbf{v}| = |(-3)(2)+(1)(-1)+(0)(3)| = |-7| = 7$

$$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{(-3)^2+1^2+0^2}\sqrt{2^2+(-1)^2+3^2} = \sqrt{10}\sqrt{14} \approx 11.832$$

(c) $|\mathbf{u}\cdot\mathbf{v}| = |(0)(1)+(2)(1)+(2)(1)+(1)(1)| = 5$

$$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{0^2+2^2+2^2+1^2}\sqrt{1^2+1^2+1^2+1^2} = \sqrt{9}\sqrt{4} = 6$$

27. It is a sphere of radius 1 centered at (x_0, y_0, z_0).

True/False 3.2

(a) True; $\|2\mathbf{v}\| = 2\|\mathbf{v}\|$.

(b) True

(c) False; $\|\mathbf{0}\| = 0$.

(d) True; the vectors are $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ and $-\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$.

(e) True; $\cos\dfrac{\pi}{3} = \dfrac{1}{2} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$.

(f) False; $(\mathbf{u}\cdot\mathbf{v}) + \mathbf{w}$ does not make sense because $\mathbf{u}\cdot\mathbf{v}$ is a scalar. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})$ is a meaningful expression.

(g) False; for example, if $\mathbf{u} = (1, 1)$, $\mathbf{v} = (-1, 1)$, and $\mathbf{w} = (1, -1)$, then $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w} = 0$.

(h) False; let $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (-1, 1)$, then $\mathbf{u}\cdot\mathbf{v} = (1)(-1) + (1)(1) = 0$. $\mathbf{u}\cdot\mathbf{v} = 0$ indicates that the angle between $\mathbf{u}$ and $\mathbf{v}$ is 90°.

(i) True; if $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ then $u_1, u_2 > 0$ and $v_1, v_2 < 0$ so $u_1v_1, u_2v_2 < 0$, hence $u_1v_1 + u_2v_2 < 0$.

(j) True; use the triangle inequality twice.

$$\|\mathbf{u}+\mathbf{v}+\mathbf{w}\| = \|(\mathbf{u}+\mathbf{v})+\mathbf{w}\| \le \|\mathbf{u}+\mathbf{v}\| + \|\mathbf{w}\|$$
$$\|\mathbf{u}+\mathbf{v}\| + \|\mathbf{w}\| \le \|\mathbf{u}\| + \|\mathbf{v}\| + \|\mathbf{w}\|$$

Section 3.3

Exercise Set 3.3

1. (a) $\mathbf{u}\cdot\mathbf{v} = (6)(2) + (1)(0) + (4)(-3) = 0$, so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(b) $\mathbf{u}\cdot\mathbf{v} = (0)(1) + (0)(1) + (-1)(1) = -1 \ne 0$, so $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal.

(c) $\mathbf{u}\cdot\mathbf{v} = (-6)(3) + (0)(1) + (4)(6) = 6 \ne 0$, so $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal.

(d) $\mathbf{u}\cdot\mathbf{v} = (2)(5) + (4)(3) + (-8)(7) = -34 \ne 0$, so $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal.

3. (a) $\mathbf{v}_1\cdot\mathbf{v}_2 = (2)(3)+(3)(2) = 12 \ne 0$
The vectors do not form an orthogonal set.

(b) $\mathbf{v}_1\cdot\mathbf{v}_2 = (-1)(1)+(1)(1) = 0$
The vectors form an orthogonal set.

(c) $\mathbf{v}_1\cdot\mathbf{v}_2 = (-2)(1)+(1)(0)+(1)(2) = 0$
$\mathbf{v}_1\cdot\mathbf{v}_3 = (-2)(-2)+(1)(-5)+(1)(1) = 0$
$\mathbf{v}_2\cdot\mathbf{v}_3 = (1)(-2)+(0)(-5)+(2)(1) = 0$
The vectors form an orthogonal set.

(d) $\mathbf{v}_1\cdot\mathbf{v}_2 = (-3)(1)+(4)(2)+(-1)(5) = 0$
$\mathbf{v}_1\cdot\mathbf{v}_3 = (-3)(4)+(4)(-3)+(-1)(0) = -24$
Since $\mathbf{v}_1\cdot\mathbf{v}_3 \ne 0$, the vectors do not form an orthogonal set.

5. Let $\mathbf{x} = (x_1, x_2, x_3)$.
$\mathbf{u}\cdot\mathbf{x} = (1)(x_1)+(0)(x_2)+(1)(x_3) = x_1 + x_3$
$\mathbf{v}\cdot\mathbf{x} = (0)(x_1)+(1)(x_2)+(1)(x_3) = x_2 + x_3$
Thus $x_1 = x_2 = -x_3$. One vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{x} = (1, 1, -1)$.

$$\frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{(1, 1, -1)}{\sqrt{1^2+1^2+(-1)^2}} = \frac{(1, 1, -1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

The possible vectors are $\pm\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}\right)$.

7. $\overrightarrow{AB} = (-2-1, 0-1, 3-1) = (-3, -1, 2)$

$\overrightarrow{BC} = (-3-(-2), -1-0, 1-3) = (-1, -1, -2)$

$\overrightarrow{CA} = (1-(-3), 1-(-1), 1-1) = (4, 2, 0)$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = (-3)(-1)+(-1)(-1)+(2)(-2) = 0$

Since $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are orthogonal and the points form the vertices of a right triangle.

9. $-2(x-(-1)) + 1(y-3) + (-1)(z-(-2)) = 0$
$-2(x+1) + (y-3) - (z+2) = 0$

11. $0(x-2) + 0(y-0) + 2(z-0) = 0$
$2z = 0$

13. A normal to $4x - y + 2z = 5$ is $\mathbf{n}_1 = (4, -1, 2)$.
A normal to $7x - 3y + 4z = 8$ is $\mathbf{n}_2 = (7, -3, 4)$.
Since $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel, the planes are not parallel.

15. $2y = 8x - 4z + 5 \Rightarrow 8x - 2y - 4z = -5$
A normal to the plane is $\mathbf{n}_1 = (8, -2, -4)$.

$x = \frac{1}{2}z + \frac{1}{4}y \;\Rightarrow\; x - \frac{1}{4}y - \frac{1}{2}z = 0$

A normal to the plane is $\mathbf{n}_2 = \left(1, -\frac{1}{4}, -\frac{1}{2}\right)$.

Since $\mathbf{n}_1 = 8\mathbf{n}_2$, the planes are parallel.

17. A normal to $3x - y + z - 4 = 0$ is $\mathbf{n}_1 = (3, -1, 1)$ and a normal to $x + 2z = -1$ is $\mathbf{n}_2 = (1, 0, 2)$.

$\mathbf{n}_1 \cdot \mathbf{n}_2 = (3)(1) + (-1)(0) + (1)(2) = 5$

Since $\mathbf{n}_1$ and $\mathbf{n}_2$ are not orthogonal, the planes are not perpendicular.

19. (a)
$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \left\|\frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}\right\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|^2}\|\mathbf{a}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|} = \frac{|(1)(-4)+(-2)(-3)|}{\sqrt{(-4)^2+(-3)^2}} = \frac{2}{\sqrt{25}} = \frac{2}{5}$$

(b)
$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|} = \frac{|(3)(2)+(0)(3)+(4)(3)|}{\sqrt{2^2+3^2+3^2}} = \frac{18}{\sqrt{22}}$$

21. $\mathbf{u} \cdot \mathbf{a} = (6)(3) + (2)(-9) = 0$
The vector component of **u** along **a** is

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{0}{\|\mathbf{a}\|^2}(3, -9) = (0, 0).$$

The vector component of **u** orthogonal to **a** is
$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (6, 2) - (0, 0) = (6, 2)$.

23. $\mathbf{u} \cdot \mathbf{a} = (3)(1) + (1)(0) + (-7)(5) = -32$
$\|\mathbf{a}\|^2 = 1^2 + 0^2 + 5^2 = 26$
The vector component of **u** along **a** is

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{-32}{26}(1, 0, 5) = \left(-\frac{16}{13}, 0, -\frac{80}{13}\right).$$

The vector component of **u** orthogonal to **a** is

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (3, 1, -7) - \left(-\frac{16}{13}, 0, -\frac{80}{13}\right) = \left(\frac{55}{13}, 1, -\frac{11}{13}\right).$$

25. $\mathbf{u} \cdot \mathbf{a} = (1)(0) + (1)(2) + (1)(-1) = 1$

$\|\mathbf{a}\|^2 = 0^2 + 2^2 + (-1)^2 = 5$

The vector component of **u** along **a** is

$$\begin{aligned}\text{proj}_{\mathbf{a}}\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \\ &= \frac{1}{5}(0, 2, -1) \\ &= \left(0, \frac{2}{5}, -\frac{1}{5}\right).\end{aligned}$$

The vector component of **u** orthogonal to **a** is

$$\begin{aligned}\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} &= (1, 1, 1) - \left(0, \frac{2}{5}, -\frac{1}{5}\right) \\ &= \left(1, \frac{3}{5}, \frac{6}{5}\right).\end{aligned}$$

27. $\mathbf{u} \cdot \mathbf{a} = (2)(4) + (1)(-4) + (1)(2) + (2)(-2) = 2$

$\|\mathbf{a}\|^2 = 4^2 + (-4)^2 + 2^2 + (-2)^2 = 40$

The vector component of **u** along **a** is

$$\begin{aligned}\text{proj}_{\mathbf{a}}\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \\ &= \frac{2}{40}(4, -4, 2, -2) \\ &= \left(\frac{1}{5}, -\frac{1}{5}, \frac{1}{10}, -\frac{1}{10}\right).\end{aligned}$$

The vector component of **u** orthogonal to **a** is

$$\begin{aligned}\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} &= (2, 1, 1, 2) - \left(\frac{1}{5}, -\frac{1}{5}, \frac{1}{10}, -\frac{1}{10}\right) \\ &= \left(\frac{9}{5}, \frac{6}{5}, \frac{9}{10}, \frac{21}{10}\right).\end{aligned}$$

29. $$\begin{aligned}D &= \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}} \\ &= \frac{|4(-3) + 3(1) + 4|}{\sqrt{4^2+3^2}} \\ &= \frac{|-5|}{\sqrt{25}} \\ &= 1\end{aligned}$$

31. The line is $4x + y - 2 = 0$.

$$\begin{aligned}D &= \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}} \\ &= \frac{|4(2) + (1)(-5) - 2|}{\sqrt{4^2+1^2}} \\ &= \frac{1}{\sqrt{17}}\end{aligned}$$

33. The plane is $x + 2y - 2z - 4 = 0$.

$$\begin{aligned}D &= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|(1)(3) + 2(1) - 2(-2) - 4|}{\sqrt{1^2+2^2+(-2)^2}} \\ &= \frac{|5|}{\sqrt{9}} \\ &= \frac{5}{3}\end{aligned}$$

35. The plane is $2x + 3y - 4z - 1 = 0$.

$$\begin{aligned}D &= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|2(-1) + 3(2) - 4(1) - 1|}{\sqrt{2^2+3^2+(-4)^2}} \\ &= \frac{|-1|}{\sqrt{29}} \\ &= \frac{1}{\sqrt{29}}\end{aligned}$$

37. A point on $2x - y - z = 5$ is $P_0(0, -5, 0)$. The distance between P_0 and the plane $-4x + 2y + 2z = 12$ is

$$\begin{aligned}D &= \frac{|-4(0) + 2(-5) + 2(0) - 12|}{\sqrt{(-4)^2+2^2+2^2}} \\ &= \frac{|-22|}{\sqrt{24}} \\ &= \frac{22}{2\sqrt{6}} \\ &= \frac{11}{\sqrt{6}}.\end{aligned}$$

39. Note that the equation of the second plane is -2 times the equation of the first plane, so the planes coincide. The distance between the planes is 0.

41. **(a)** α is the angle between **v** and **i**, so

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\|\mathbf{i}\|} = \frac{(a)(1)+(b)(0)+(c)(0)}{\|\mathbf{v}\|\cdot 1} = \frac{a}{\|\mathbf{v}\|}.$$

(b) Similar to part (a), $\cos\beta = \frac{b}{\|\mathbf{v}\|}$ and $\cos\gamma = \frac{c}{\|\mathbf{v}\|}$.

(c) $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{a}{\|\mathbf{v}\|}, \frac{b}{\|\mathbf{v}\|}, \frac{c}{\|\mathbf{v}\|}\right) = (\cos\alpha, \cos\beta, \cos\gamma)$

(d) Since $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector, its magnitude is 1, so $\sqrt{\cos^2\alpha+\cos^2\beta+\cos^2\gamma} = 1$ or $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$.

43.
$$\begin{aligned}\mathbf{v}\cdot(k_1\mathbf{w}_1+k_2\mathbf{w}_2) &= \mathbf{v}\cdot(k_1\mathbf{w}_1)+\mathbf{v}\cdot(k_2\mathbf{w}_2)\\ &= k_1(\mathbf{v}\cdot\mathbf{w}_1)+k_2(\mathbf{v}\cdot\mathbf{w}_2)\\ &= k_1(0)+k_2(0)\\ &= 0\end{aligned}$$

True/False 3.3

(a) True; the zero vector is orthogonal to all vectors.

(b) True; $(k\mathbf{u})\cdot(m\mathbf{v}) = km(\mathbf{u}\cdot\mathbf{v}) = km(0) = 0$.

(c) True; the orthogonal projection of **u** on **a** has the same direction as **a** while the vector component of **u** orthogonal to **a** is orthogonal (perpendicular) to **a**.

(d) True; since $\text{proj}_{\mathbf{b}}\mathbf{u}$ has the same direction as **b**.

(e) True; if **v** has the same direction as **a**, then $\text{proj}_{\mathbf{a}}\mathbf{v} = \mathbf{v}$.

(f) False; for **a** = (1, 0), **u** = (1, 10) and **v** = (1, 7), $\text{proj}_{\mathbf{a}}\mathbf{u} = \text{proj}_{\mathbf{a}}\mathbf{v}$.

(g) False; $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\|+\|\mathbf{v}\|$.

Section 3.4

Exercise Set 3.4

1. The vector equation is $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$.
$(x, y) = (-4, 1) + t(0, -8)$
Equating components gives the parametric equations.
$x = -4,\ y = 1 - 8t$

3. Since the given point is the origin, the vector equation is $\mathbf{x} = t\mathbf{v}$.
$(x, y, z) = t(-3, 0, 1)$
Equating components gives the parametric equations.
$x = -3t,\ y = 0,\ z = t$

5. For $t = 0$, the point is (3, −6). The coefficients of t give the parallel vector (−5, −1).

7. For $t = 0$, the point is (4, 6).
$\mathbf{x} = (4 - 4t, 6 - 6t) + (-2t, 0) = (4 - 6t, 6 - 6t)$
The coefficients of t give the parallel vector (−6, −6).

9. The vector equation is $\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$.
$(x, y, z) = (-3, 1, 0) + t_1(0, -3, 6) + t_2(-5, 1, 2)$
Equating components gives the parametric equations.
$x = -3 - 5t_2,\quad y = 1 - 3t_1 + t_2,\quad z = 6t_1 + 2t_2$

11. The vector equation is $\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$.
$(x, y, z) = (-1, 1, 4) + t_1(6, -1, 0) + t_2(-1, 3, 1)$
Equating components gives the parametric equations.
$x = -1 + 6t_1 - t_2,\quad y = 1 - t_1 + 3t_2,\quad z = 4 + t_2$

13. One vector orthogonal to **v** is **w** = (3, 2). Using **w**, the vector equation is $(x, y) = t(3, 2)$ and the parametric equations are $x = 3t$, $y = 2t$.

15. Two possible vectors that are orthogonal to **v** are $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (5, 0, 4)$. Using $\mathbf{v}_1$ and $\mathbf{v}_2$, the vector equation is $(x, y, z) = t_1(0, 1, 0) + t_2(5, 0, 4)$ and the parametric equations are $x = 5t_2,\quad y = t_1,\quad z = 4t_2$.

17. The augmented matrix of the system is $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix}$ which has reduced row echelon form $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The solution of the system is $x_1 = -r - s,\ x_2 = r,\ x_3 = s,$ or $\mathbf{x} = (-r - s, r, s)$ in vector form.

Since the row vectors of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ are all multiples of (1, 1, 1), only $\mathbf{r}_1$ needs to be considered.

$$\begin{aligned} \mathbf{r}_1 \cdot \mathbf{x} &= (1, 1, 1) \cdot (-r - s,\ r,\ s) \\ &= 1(-r - s) + 1(r) + 1(s) \\ &= 0 \end{aligned}$$

19. The augmented matrix of the system is $\begin{bmatrix} 1 & 5 & 1 & 2 & -1 & 0 \\ 1 & -2 & -1 & 3 & 2 & 0 \end{bmatrix}$ which has reduced row echelon form

$\begin{bmatrix} 1 & 0 & -\frac{3}{7} & \frac{19}{7} & \frac{8}{7} & 0 \\ 0 & 1 & \frac{2}{7} & -\frac{1}{7} & -\frac{3}{7} & 0 \end{bmatrix}$.

The solution of the system is $x_1 = \frac{3}{7}r - \frac{19}{7}s - \frac{8}{7}t,\ x_2 = -\frac{2}{7}r + \frac{1}{7}s + \frac{3}{7}t,\ x_3 = r,\ x_4 = s,\ x_5 = t,$ or

$\mathbf{x} = \left(\frac{3}{7}r - \frac{19}{7}s - \frac{8}{7}t,\ -\frac{2}{7}r + \frac{1}{7}s + \frac{3}{7}t,\ r,\ s,\ t\right)$ in vector form.

For this system, $\mathbf{r}_1 = (1, 5, 1, 2, -1)$ and $\mathbf{r}_2 = (1, -2, -1, 3, 2)$.

$$\begin{aligned} \mathbf{r}_1 \cdot \mathbf{x} &= 1\left(\frac{3}{7}r - \frac{19}{7}s - \frac{8}{7}t\right) + 5\left(-\frac{2}{7}r + \frac{1}{7}s + \frac{3}{7}t\right) + 1(r) + 2(s) - 1(t) \\ &= \frac{3}{7}r - \frac{19}{7}s - \frac{8}{7}t - \frac{10}{7}r + \frac{5}{7}s + \frac{15}{7}t + r + 2s - t \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{r}_2 \cdot \mathbf{x} &= 1\left(\frac{3}{7}r - \frac{19}{7}s - \frac{8}{7}t\right) - 2\left(-\frac{2}{7}r + \frac{1}{7}s + \frac{3}{7}t\right) - 1(r) + 3(s) + 2(t) \\ &= \frac{3}{7}r - \frac{19}{7}s - \frac{8}{7}t + \frac{4}{7}r - \frac{2}{7}s - \frac{6}{7}t - r + 3s + 2t \\ &= 0 \end{aligned}$$

21. **(a)** A particular solution of $x + y + z = 1$ is (1, 0, 0).
The general solution of $x + y + z = 0$ is $x = -s - t$, $y = s$, $z = t$, which is $s(-1, 1, 0) + t(-1, 0, 1)$ in vector form.
The equation can be represented by $(1, 0, 0) + s(-1, 1, 0) + t(-1, 0, 1)$.

(b) Geometrically, the form of the equation indicates that it is a plane in R^3, passing through the point $P(1, 0, 0)$ and parallel to the vectors (−1, 1, 0) and (−1, 0, 1).

23. **(a)** If $\mathbf{x} = (x, y, z)$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a} \cdot \mathbf{x} = x + y + z = 0$ and $\mathbf{b} \cdot \mathbf{x} = -2x + 3y = 0$.
The system is $\begin{aligned} x + y + z &= 0 \\ -2x + 3y &= 0 \end{aligned}$.

(b) The reduced row echelon form of $\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 3 & 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & \frac{3}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \end{bmatrix}$ so the solution will require one parameter. Since only one parameter is required, the solution space is a line through the origin in R^3.

(c) From part (b), the solution of the system is $x = -\frac{3}{5}t,\ \ y = -\frac{2}{5}t,\ \ z = t$, or $\mathbf{x} = \left(-\frac{3}{5}t, -\frac{2}{5}t, t\right)$ in vector form. Here, $\mathbf{r}_1 = (1, 1, 1)$ and $\mathbf{r}_2 = (-2, 3, 0)$.

$$\mathbf{r}_1 \cdot \mathbf{x} = 1\left(-\frac{3}{5}t\right) + 1\left(-\frac{2}{5}t\right) + 1(t) = 0$$

$$\mathbf{r}_2 \cdot \mathbf{x} = -2\left(-\frac{3}{5}t\right) + 3\left(-\frac{2}{5}t\right) + 0(t) = 0$$

25. (a) The reduced row echelon form of

$$\begin{bmatrix} 3 & 2 & -1 & 0 \\ 6 & 4 & -2 & 0 \\ -3 & -2 & 1 & 0 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution of the system is $x_1 = -\frac{2}{3}s + \frac{1}{3}t,\ \ x_2 = s,\ \ x_3 = t.$

(b) Since the second and third rows of $\begin{bmatrix} 3 & 2 & -1 & 2 \\ 6 & 4 & -2 & 4 \\ -3 & -2 & 1 & -2 \end{bmatrix}$ are scalar multiples of the first, it suffices to show that $x_1 = 1$, $x_2 = 0,\ \ x_3 = 1$ is a solution of $3x_1 + 2x_2 - x_3 = 2$.

$3(1) + 2(0) - 1 = 3 - 1 = 2.$

(c) In vector form, add (1, 0, 1) to $\left(-\frac{2}{3}s + \frac{1}{3}t,\ s,\ t\right)$ to get

$x_1 = 1 - \frac{2}{3}s + \frac{1}{3}t,\ x_2 = s,\ x_3 = 1 + t.$

27. The reduced row echelon form of

$$\begin{bmatrix} 3 & 4 & 1 & 2 & 3 \\ 6 & 8 & 2 & 5 & 7 \\ 9 & 12 & 3 & 10 & 13 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A general solution of the system is $x_1 = \frac{1}{3} - \frac{4}{3}s - \frac{1}{3}t,\ \ x_2 = s,\ \ x_3 = t,\ \ x_4 = 1.$

From the general solution of the nonhomogeneous system, a general solution of the homogeneous system is $x_1 = -\frac{4}{3}s - \frac{1}{3}t$, $x_2 = s,\ \ x_3 = t,\ \ x_4 = 0$ and a particular solution of the nonhomogeneous system is $x_1 = \frac{1}{3}$, $x_2 = 0,\ \ x_3 = 0,\ \ x_4 = 1.$

True/False 3.4

(a) True; the vector equation is then $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ where $\mathbf{x}_0$ is the given point and $\mathbf{v}$ is the given vector.

(b) False; two non collinear vectors that are parallel to the plane are needed.

(c) True; the equation of the line is $\mathbf{x} = t\mathbf{v}$ where $\mathbf{v}$ is any nonzero vector on the line.

(d) True; if $\mathbf{b} = \mathbf{0}$, then the statement is true by Theorem 3.4.3. If $\mathbf{b} \neq \mathbf{0}$, so that there is a nonzero entry, say b_i of $\mathbf{b}$, and if $\mathbf{x}_0$ is a solution vector of the system, then by matrix multiplication, $\mathbf{r}_i \cdot \mathbf{x}_0 = b_i$ where $\mathbf{r}_i$ is the ith row vector of A.

(e) False; a particular solution of $A\mathbf{x} = \mathbf{b}$ must be used to obtain the general solution of $A\mathbf{x} = \mathbf{b}$ from the general solution of $A\mathbf{x} = \mathbf{0}$.

(f) True; $A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

Section 3.5

Exercise Set 3.5

1. (a)

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= (0, 2, -3) \times (2, 6, 7) \\ &= \left(\begin{vmatrix} 2 & -3 \\ 6 & 7 \end{vmatrix}, -\begin{vmatrix} 0 & -3 \\ 2 & 7 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 2 & 6 \end{vmatrix} \right) \\ &= (14 + 18, -(0 + 6), 0 - 4) \\ &= (32, -6, -4) \end{aligned}$$

(b)

$$\begin{aligned} &\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \\ &= (3, 2, -1) \times (32, -6, -4) \\ &= \left(\begin{vmatrix} 2 & -6 \\ -1 & -4 \end{vmatrix}, -\begin{vmatrix} 3 & -1 \\ 32 & -4 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 32 & -6 \end{vmatrix} \right) \\ &= (-8 - 6, -(-12 + 32), -18 - 64) \\ &= (-14, -20, -82) \end{aligned}$$

(c) $\mathbf{u}\times\mathbf{v} = (3, 2, -1)\times(0, 2, -3)$
$$= \left(\begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix}, -\begin{vmatrix} 3 & -1 \\ 0 & -3 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix}\right)$$
$$= (-6+2, -(-9-0), 6-0)$$
$$= (-4, 9, 6)$$
$(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$
$$= (-4, 9, 6)\times(2, 6, 7)$$
$$= \left(\begin{vmatrix} 9 & 6 \\ 6 & 7 \end{vmatrix}, -\begin{vmatrix} -4 & 6 \\ 2 & 7 \end{vmatrix}, \begin{vmatrix} -4 & 9 \\ 2 & 6 \end{vmatrix}\right)$$
$$= (63-36, -(-28-12), -24-18)$$
$$= (27, 40, -42)$$

3. $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
$$\mathbf{u}\times\mathbf{v} = (-6, 4, 2)\times(3, 1, 5)$$
$$= \left(\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix}, -\begin{vmatrix} -6 & 2 \\ 3 & 5 \end{vmatrix}, \begin{vmatrix} -6 & 4 \\ 3 & 1 \end{vmatrix}\right)$$
$$= (20-2, -(-30-6), -6-12)$$
$$= (18, 36, -18)$$

5. $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
$$\mathbf{u}\times\mathbf{v} = (-2, 1, 5)\times(3, 0, -3)$$
$$= \left(\begin{vmatrix} 1 & 5 \\ 0 & -3 \end{vmatrix}, -\begin{vmatrix} -2 & 5 \\ 3 & -3 \end{vmatrix}, \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix}\right)$$
$$= (-3-0, -(6-15), 0-3)$$
$$= (-3, 9, -3)$$

7. $\mathbf{u}\times\mathbf{v} = (1, -1, 2)\times(0, 3, 1)$
$$= \left(\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix}\right)$$
$$= (-1-6, -(1-0), 3-0)$$
$$= (-7, -1, 3)$$
$$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{(-7)^2+(-1)^2+3^2} = \sqrt{59}$$
The area is $\sqrt{59}$.

9. $\mathbf{u}\times\mathbf{v} = (2, 3, 0)\times(-1, 2, -2)$
$$= \left(\begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix}, -\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}\right)$$
$$= (-6-0, -(-4-0), 4+3)$$
$$= (-6, 4, 7)$$
$$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{(-6)^2+4^2+7^2} = \sqrt{101}$$
The area is $\sqrt{101}$.

11. $\overrightarrow{P_1P_2} = (3, 2)$ and $\overrightarrow{P_3P_4} = (-3, -2)$ so the side determined by P_1 and P_2 is parallel to the side determined by P_3 and P_4, but the direction is opposite, thus P_1P_2 and P_1P_4 are adjacent sides.

$\overrightarrow{P_1P_4} = (3, 1)$

The parallelogram can be considered in R^3 as being determined by $\mathbf{u} = (3, 2, 0)$ and $\mathbf{v} = (3, 1, 0)$.
$$\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 3 & 0 \\ 3 & 0 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 3 & 1 \end{vmatrix}\right)$$
$$= (0, 0, -3)$$
$$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{0^2+0^2+(-3)^2} = 3$$
The area is 3.

13. $\overrightarrow{AB} = (1, 4)$ and $\overrightarrow{AC} = (-3, 2)$

The triangle can be considered in R^3 as being half of the parallelogram formed by $\mathbf{u} = (1, 4, 0)$ and $\mathbf{v} = (-3, 2, 0)$.
$$\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix} 4 & 0 \\ 2 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ -3 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ -3 & 2 \end{vmatrix}\right)$$
$$= (0, 0, 14)$$
$$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{0^2+0^2+14^2} = 14$$
The area of the triangle is $\frac{1}{2}\|\mathbf{u}\times\mathbf{v}\| = 7.$

15. Let $\mathbf{u} = \overrightarrow{P_1P_2} = (-1, -5, 2)$ and $\mathbf{v} = \overrightarrow{P_1P_3} = (2, 0, 3)$.
$$\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix} -5 & 2 \\ 0 & 3 \end{vmatrix}, -\begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} -1 & -5 \\ 2 & 0 \end{vmatrix}\right)$$
$$= (-15, 7, 10)$$
$$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{(-15)^2+7^2+10^2} = \sqrt{374}$$
The area of the triangle is $\frac{1}{2}\|\mathbf{u}\times\mathbf{v}\| = \frac{\sqrt{374}}{2}.$

17. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{vmatrix}$
$$= 2\begin{vmatrix} 4 & -2 \\ 2 & -4 \end{vmatrix} + 2\begin{vmatrix} -6 & 2 \\ 4 & -2 \end{vmatrix}$$
$$= 2(-12)+2(4)$$
$$= -16$$
The volume of the parallelepiped is $|-16| = 16.$

19. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} -1 & -2 & 1 \\ 3 & 0 & -2 \\ 5 & -4 & 0 \end{vmatrix}$
$= \begin{vmatrix} 3 & 0 \\ 5 & -4 \end{vmatrix} + 2\begin{vmatrix} -1 & -2 \\ 5 & -4 \end{vmatrix}$
$= -12 + 2(14)$
$= 16$

The vectors do not lie in the same plane.

21. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} -2 & 0 & 6 \\ 1 & -3 & 1 \\ -5 & -1 & 1 \end{vmatrix}$
$= -2\begin{vmatrix} -3 & 1 \\ -1 & 1 \end{vmatrix} + 6\begin{vmatrix} 1 & -3 \\ -5 & -1 \end{vmatrix}$
$= -2(-2) + 6(-16)$
$= -92$

23. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$

25. Since $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 3$, then
$\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = 3.$

(a) $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = \det\begin{bmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = -3$

(Rows 2 and 3 were interchanged.)

(b) $(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 3$

(c) $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \det\begin{bmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = 3$

(Rows 2 and 3 were interchanged, then rows 1 and 2 were interchanged.)

27. (a) Let $\mathbf{u} = \overrightarrow{AB} = (-1, 2, 2)$ and

$\mathbf{v} = \overrightarrow{AC} = (1, 1, -1).$

$\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix}, -\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix}\right)$
$= (-4, 1, -3)$

$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{(-4)^2 + 1^2 + (-3)^2} = \sqrt{26}$

The area of the triangle is $\frac{1}{2}\|\mathbf{u}\times\mathbf{v}\| = \frac{\sqrt{26}}{2}$.

(b) $\left\|\overrightarrow{AB}\right\| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3$

Let x be the length of the altitude from vertex C to side AB, then

$\frac{1}{2}x\left\|\overrightarrow{AB}\right\| = \frac{\sqrt{26}}{2}$

$x = \frac{\sqrt{26}}{\left\|\overrightarrow{AB}\right\|} = \frac{\sqrt{26}}{3}$

29. $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})$
$= (\mathbf{u}+\mathbf{v})\times\mathbf{u} - (\mathbf{u}+\mathbf{v})\times\mathbf{v}$
$= (\mathbf{u}\times\mathbf{u}) + (\mathbf{v}\times\mathbf{u}) - ((\mathbf{u}\times\mathbf{v}) + (\mathbf{v}\times\mathbf{v}))$
$= 0 + (\mathbf{v}\times\mathbf{u}) - (\mathbf{u}\times\mathbf{v}) - 0$
$= (\mathbf{v}\times\mathbf{u}) + (\mathbf{v}\times\mathbf{u})$
$= 2(\mathbf{v}\times\mathbf{u})$

37. (a) $\mathbf{a} = \overrightarrow{PQ} = (3, -1, -3)$

$\mathbf{b} = \overrightarrow{PR} = (2, -1, 1)$

$\mathbf{c} = \overrightarrow{PS} = (4, -4, 3)$

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 3 & -1 & -3 \\ 2 & -1 & 1 \\ 4 & -4 & 3 \end{vmatrix}$
$= 3\begin{vmatrix} -1 & 1 \\ -4 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & -1 \\ 4 & -4 \end{vmatrix}$
$= 3(1) + 1(2) - 3(-4)$
$= 17$

The volume is $\frac{1}{6}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = \frac{17}{6}$.

(b) $\mathbf{a} = \overrightarrow{PQ} = (1, 2, -1)$

$\mathbf{b} = \overrightarrow{PR} = (3, 4, 0)$

$\mathbf{c} = \overrightarrow{PS} = (-1, -3, 4)$

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ -1 & -3 & 4 \end{vmatrix}$
$= -1\begin{vmatrix} 3 & 4 \\ -1 & -3 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$
$= -(-5) + 4(-2)$
$= -3$

The volume is $\frac{1}{6}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = \frac{3}{6} = \frac{1}{2}$.

True/False 3.5

(a) True; for nonzero vectors **u** and **v**, $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$ will only be 0 if $\theta = 0$, i.e., if **u** and **v** are parallel.

(b) True; the cross product of two nonzero and non collinear vectors will be perpendicular to both vectors, hence normal to the plane containing the vectors.

(c) False; the scalar triple product is a scalar, not a vector.

(d) True

(e) False; for example
$(\mathbf{i}\times\mathbf{i})\times\mathbf{j} = \mathbf{0}\times\mathbf{j} = \mathbf{0}$
$\mathbf{i}\times(\mathbf{i}\times\mathbf{j}) = \mathbf{i}\times\mathbf{k} = -\mathbf{j}$

(f) False; for example, if **v** and **w** are distinct vectors that are both parallel to **u**, then $\mathbf{u}\times\mathbf{v} = \mathbf{u}\times\mathbf{w} = \mathbf{0}$, but $\mathbf{v}\neq\mathbf{w}$.

Chapter 3 Supplementary Exercises

1. (a)
$$\begin{aligned}3\mathbf{v}-2\mathbf{u} &= (9, -3, 18)-(-4, 0, 8)\\ &= (13, -3, 10)\end{aligned}$$

(b)
$$\begin{aligned}&\|\mathbf{u}+\mathbf{v}+\mathbf{w}\|\\ &= \|(-2+3+2,\ 0-1-5,\ 4+6-5)\|\\ &= \|(3, -6, 5)\|\\ &= \sqrt{3^2+(-6)^2+5^2}\\ &= \sqrt{70}\end{aligned}$$

(c) The distance between $-3\mathbf{u}$ and $\mathbf{v} + 5\mathbf{w}$ is $\|-3\mathbf{u}-(\mathbf{v}+5\mathbf{w})\| = \|-3\mathbf{u}-\mathbf{v}-5\mathbf{w}\|$.
$$\begin{aligned}&\|-3\mathbf{u}-\mathbf{v}-5\mathbf{w}\|\\ &= \|(6, 0, -12)-(3, -1, 6)-(10, -25, -25)\|\\ &= \|(-7, 26, 7)\|\\ &= \sqrt{(-7)^2+26^2+7^2}\\ &= \sqrt{774}\end{aligned}$$

(d) $\mathbf{u}\cdot\mathbf{w} = (-2)(2) + (0)(-5) + (4)(-5) = -24$
$\|\mathbf{w}\|^2 = 2^2+(-5)^2+(-5)^2 = 54$
$$\begin{aligned}\text{proj}_{\mathbf{w}}\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}\\ &= \frac{-24}{54}(2, -5, -5)\\ &= -\frac{12}{27}(2, -5, -5)\end{aligned}$$

(e)
$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \begin{vmatrix}-2 & 0 & 4\\ 3 & -1 & 6\\ 2 & -5 & -5\end{vmatrix}\\ &= -2\begin{vmatrix}-1 & 6\\ -5 & -5\end{vmatrix}+4\begin{vmatrix}3 & -1\\ 2 & -5\end{vmatrix}\\ &= -2(35)+4(-13)\\ &= -122\end{aligned}$$

(f)
$$\begin{aligned}-5\mathbf{v}+\mathbf{w} &= (-15, 5, -30)+(2, -5, -5)\\ &= (-13, 0, -35)\end{aligned}$$
$$\begin{aligned}&(\mathbf{u}\cdot\mathbf{v})\mathbf{w}\\ &= ((-2)(3)+(0)(-1)+(4)(6))(2, -5, -5)\\ &= 18(2, -5, -5)\\ &= (36, -90, -90)\end{aligned}$$
$$\begin{aligned}&(-5\mathbf{v}+\mathbf{w})\times(\mathbf{u}\cdot\mathbf{v})\mathbf{w}\\ &= \left(\begin{vmatrix}0 & -35\\ -90 & -90\end{vmatrix}, -\begin{vmatrix}-13 & -35\\ 36 & -90\end{vmatrix}, \begin{vmatrix}-13 & 0\\ 36 & -90\end{vmatrix}\right)\\ &= (-3150, -2430, 1170)\end{aligned}$$

3. (a)
$$\begin{aligned}3\mathbf{v}-2\mathbf{u} &= (-9, 0, 24, 0)-(-4, 12, 4, 2)\\ &= (-5, -12, 20, -2)\end{aligned}$$

(b)
$$\begin{aligned}&\|\mathbf{u}+\mathbf{v}+\mathbf{w}\|\\ &= \|(-2-3+9,\ 6+0+1,\ 2+8-6,\ 1+0-6)\|\\ &= \|(4, 7, 4, -5)\|\\ &= \sqrt{4^2+7^2+4^2+(-5)^2}\\ &= \sqrt{106}\end{aligned}$$

(c) The distance between $-3\mathbf{u}$ and $\mathbf{v} + 5\mathbf{w}$ is $\|-3\mathbf{u}-(\mathbf{v}+5\mathbf{w})\| = \|-3\mathbf{u}-\mathbf{v}-5\mathbf{w}\|$.

$$\begin{aligned}\|-3\mathbf{u}-\mathbf{v}-5\mathbf{w}\| &= \|(6, -18, -6, -3)-(-3, 0, 8, 0)-(45, 5, -30, -30)\| \\ &= \|(-36, -23, 16, 27)\| \\ &= \sqrt{(-36)^2+(-23)^2+16^2+27^2} \\ &= \sqrt{2810}\end{aligned}$$

(d) $$\begin{aligned}\mathbf{u}\cdot\mathbf{w} &= (-2)(9)+(6)(1)+(2)(-6)+(1)(-6) \\ &= -30\end{aligned}$$

$$\|\mathbf{w}\|^2 = 9^2+1^2+(-6)^2+(-6)^2 = 154$$

$$\begin{aligned}\text{proj}_{\mathbf{w}}\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} \\ &= \frac{-30}{154}(9, 1, -6, -6) \\ &= -\frac{15}{77}(9, 1, -6, -6)\end{aligned}$$

5. $\mathbf{u} = (-32, -1, 19)$, $\mathbf{v} = (3, -1, 5)$, $\mathbf{w} = (1, 6, 2)$
$\mathbf{u}\cdot\mathbf{v} = (-32)(3) + (-1)(-1) + (19)(5) = 0$
$\mathbf{u}\cdot\mathbf{w} = (-32)(1) + (-1)(6) + (19)(2) = 0$
$\mathbf{v}\cdot\mathbf{w} = (3)(1) + (-1)(6) + (5)(2) = 7$
Since $\mathbf{v}\cdot\mathbf{w} \neq 0$, the vectors do not form an orthogonal set.

7. **(a)** The set of all such vectors is the line through the origin which is perpendicular to the given vector.

(b) The set of all such vectors is the plane through the origin which is perpendicular to the given vector.

(c) The only vector in R^2 that can be orthogonal to two non collinear vectors is $\mathbf{0}$; the set is $\{\mathbf{0}\}$, the origin.

(d) The set of all such vectors is the line through the origin which is perpendicular to the plane containing the two given vectors.

9. True; $$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} \\ &= \|\mathbf{u}\|^2+\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}+\|\mathbf{v}\|^2\end{aligned}$$

Thus, if $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2+\|\mathbf{v}\|^2$, $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u} = 0$, so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

11. Let S be $S(-1, s_2, s_3)$. Then $\mathbf{u} = \overrightarrow{PQ} = (3, 1, -2)$ and $\mathbf{v} = \overrightarrow{RS} = (-6, s_2-1, s_3-1)$. $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \left(\begin{vmatrix}1 & -2\\ s_2-1 & s_3-1\end{vmatrix}, -\begin{vmatrix}3 & -2\\ -6 & s_3-1\end{vmatrix}, \begin{vmatrix}3 & 1\\ -6 & s_2-1\end{vmatrix}\right) \\ &= (s_3-1+2(s_2-1), -(3(s_3-1)-12), 3(s_2-1)+6) \\ &= (2s_2+s_3-3, -3s_3+15, 3s_2+3)\end{aligned}$$

If $\mathbf{u}\times\mathbf{v} = \mathbf{0}$, then from the second and third components, $s_2 = -1$ and $s_3 = 5$, which also causes the first component to be 0. The point is $S(-1, -1, 5)$.

13. $\mathbf{u} = \overrightarrow{PQ} = (3, 1, -2),\ \mathbf{v} = \overrightarrow{PR} = (2, 2, -3)$

$$\begin{aligned}\cos\theta &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\\ &= \frac{(3)(2)+(1)(2)+(-2)(-3)}{\sqrt{3^2+1^2+(-2)^2}\sqrt{2^2+2^2+(-3)^2}}\\ &= \frac{14}{\sqrt{14}\sqrt{17}}\\ &= \sqrt{\frac{14}{17}}\end{aligned}$$

15. The plane is $5x - 3y + z + 4 = 0$.

$$D = \frac{|5(-3)-3(1)+3+4|}{\sqrt{5^2+(-3)^2+1^2}} = \frac{11}{\sqrt{35}}$$

17. The plane will contain $\mathbf{u} = \overrightarrow{PQ} = (1, -2, -2)$ and $\mathbf{v} = \overrightarrow{PR} = (5, -1, -5)$.

Using $P(-2, 1, 3)$, the vector equation is

$$(x, y, z) = (-2, 1, 3) + t_1(1, -2, -2) + t_2(5, -1, -5).$$

Equating components gives the parametric equations $x = -2 + t_1 + 5t_2$, $y = 1 - 2t_1 - t_2$, $z = 3 - 2t_1 - 5t_2$.

19. The vector equation is $(x, y) = (0, -3) + t(8, -1)$. Equating components gives the parametric equations $x = 8t$, $y = -3 - t$.

21. One point on the line is $(0, -5)$. Since the slope of the line is $m = \frac{3}{1}$, the vector $(1, 3)$ is parallel to the line. Using this point and vector gives the vector equation $(x, y) = (0, -5) + t(1, 3)$. Equating components gives the parametric equations $x = t$, $y = -5 + 3t$.

23. From the vector equation, $(-1, 5, 6)$ is a point on the plane and $(0, -1, 3) \times (2, -1, 0)$ will be a normal to the plane.

$$\begin{aligned}&(0, -1, 3)\times(2, -1, 0)\\ &= \left(\begin{vmatrix}-1 & 3\\ -1 & 0\end{vmatrix}, -\begin{vmatrix}0 & 3\\ 2 & 0\end{vmatrix}, \begin{vmatrix}0 & -1\\ 2 & -1\end{vmatrix}\right)\\ &= (3, 6, 2)\end{aligned}$$

A point-normal equation for the plane is $3(x + 1) + 6(y - 5) + 2(z - 6) = 0$.

25. Two vectors in the plane are

$\mathbf{u} = \overrightarrow{PQ} = (-10, 4, -1)$ and

$\mathbf{v} = \overrightarrow{PR} = (-9, 6, -6)$.

A normal to the plane is $\mathbf{u} \times \mathbf{v}$.

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \left(\begin{vmatrix}4 & -1\\ 6 & -6\end{vmatrix}, -\begin{vmatrix}-10 & -1\\ -9 & -6\end{vmatrix}, \begin{vmatrix}-10 & 4\\ -9 & 6\end{vmatrix}\right)\\ &= (-18, -51, -24)\end{aligned}$$

Using point $P(9, 0, 4)$, a point-normal equation for the plane is $-18(x - 9) - 51y - 24(z - 4) = 0$.

29. The equation represents a plane perpendicular to the xy-plane which intersects the xy-plane along the line $Ax + By = 0$.

Chapter 4

General Vector Spaces

Section 4.1

Exercise Set 4.1

1. **(a)** $\mathbf{u}+\mathbf{v} = (-1, 2)+(3, 4)$
$= (-1+3, 2+4)$
$= (2, 6)$

$3\mathbf{u} = 3(-1, 2) = (0, 6)$

(b) The sum of any two real numbers is a real number. The product of two real numbers is a real number and 0 is a real number.

(c) Axioms 1−5 hold in V because they also hold in R^2.

(e) Let $\mathbf{u} = (u_1, u_2)$ with $u_1 \neq 0$. Then $1\mathbf{u} = 1(u_1, u_2) = (0, u_2) \neq \mathbf{u}$.

3. The set is a vector space with the given operations.

5. The set is not a vector space. Axiom 5 fails to hold because of the restriction that $x \geq 0$. Axiom 6 fails to hold for $k < 0$.

7. The set is not a vector space. Axiom 8 fails to hold because $(k+m)^2 \neq k^2 + m^2$.

9. The set is a vector space with the given operations.

11. The set is a vector space with the given operations.

23.

$\mathbf{u}+\mathbf{w} = \mathbf{v}+\mathbf{w}$	Hypothesis
$(\mathbf{u}+\mathbf{w})+(-\mathbf{w}) = (\mathbf{v}+\mathbf{w})+(-\mathbf{w})$	Add $-\mathbf{w}$ to both sides.
$\mathbf{u}+[\mathbf{w}+(-\mathbf{w})] = \mathbf{v}+[\mathbf{w}+(-\mathbf{w})]$	Axiom 3
$\mathbf{u}+\mathbf{0} = \mathbf{v}+\mathbf{0}$	Axiom 5
$\mathbf{u} = \mathbf{v}$	Axiom 4

25. (1) Axiom 7
(2) Axiom 4
(3) Axiom 5
(4) Axiom 1
(5) Axiom 3
(6) Axiom 5
(7) Axiom 4

True/False 4.1

(a) False; vectors are not restricted to being directed line segments.

(b) False; vectors are not restricted to being n-tuples of real numbers.

(c) True

(d) False; if a vector space V had exactly two elements, one of them would necessarily be the zero vector $\mathbf{0}$. Call the other vector $\mathbf{u}$. Then $\mathbf{0} + \mathbf{0} = \mathbf{0}$, $\mathbf{0} + \mathbf{u} = \mathbf{u}$, and $\mathbf{u} + \mathbf{0} = \mathbf{u}$. Since $-\mathbf{u}$ must exist and $-\mathbf{u} \neq \mathbf{0}$, then $-\mathbf{u} = \mathbf{u}$ and $\mathbf{u} + \mathbf{u} = \mathbf{0}$. Consider the scalar $\frac{1}{2}$. Since $\frac{1}{2}\mathbf{u}$ would be an element of V, $\frac{1}{2}\mathbf{u} = \mathbf{0}$ or $\frac{1}{2}\mathbf{u} = \mathbf{u}$. If $\frac{1}{2}\mathbf{u} = \mathbf{0}$, then

$$\mathbf{u} = 1\mathbf{u} = \left(\frac{1}{2}+\frac{1}{2}\right)\mathbf{u} = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{u} = \mathbf{0}+\mathbf{0} = \mathbf{0}.$$

If $\frac{1}{2}\mathbf{u} = \mathbf{u}$, then

$$\mathbf{u} = 1\mathbf{u} = \left(\frac{1}{2}+\frac{1}{2}\right)\mathbf{u} = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{u} = \mathbf{u}+\mathbf{u} = \mathbf{0}.$$

Either way we get $\mathbf{u} = \mathbf{0}$ which contradicts our assumption that we had exactly two elements.

(e) False; the zero vector would be $\mathbf{0} = 0 + 0x$ which does not have degree exactly 1.

Section 4.2

Exercise Set 4.2

1. **(a)** This is a subspace of R^3.

(b) This is not a subspace of R^3, since $(a_1, 1, 1) + (a_2, 1, 1) = (a_1 + a_2, 2, 2)$ which is not in the set.

(c) This is a subspace of R^3.

(d) This is not a subspace of R^3, since for $k \neq 1$ $k(a + c + 1) \neq ka + kc + 1$, so $k(a, b, c)$ is not in the set.

(e) This is a subspace of R^3.

3. **(a)** This is a subspace of P_3.

(b) This is a subspace of P_3.

(c) This is not a subspace of P_3 since $k(a_0 + a_1x + a_2x^2 + a_3x^3)$ is not in the set for all noninteger values of k.

(d) This is a subspace of P_3.

5. **(a)** This is a subspace of R^∞.

(b) This is not a subspace of R^∞, since for $k \neq 1$, $k\mathbf{v}$ is not in the set.

(c) This is a subspace of R^∞.

(d) This is a subspace of R^∞.

7. Consider $k_1\mathbf{u} + k_2\mathbf{v} = (a, b, c)$. Thin $(0k_1 + k_2, -2k_1 + 3k_2, 2k_1 - k_2) = (a, b, c)$. Equating components gives the system

$$\begin{aligned} k_2 &= a \\ -2k_1 + 3k_2 &= b \\ 2k_1 - k_2 &= c \end{aligned}$$ or $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 2 & -1 \end{bmatrix}$,

$\mathbf{x} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

The system can be solved simultaneously by reducing the matrix $\left[\begin{array}{rr|r|r|r|r} 0 & 1 & 2 & 3 & 0 & 0 \\ -2 & 3 & 2 & 1 & 4 & 0 \\ 2 & -1 & 2 & 5 & 5 & 0 \end{array}\right]$,

which reduces to $\left[\begin{array}{rr|r|r|r|r} 1 & 0 & 2 & 4 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$. The results can be read from the reduced matrix.

(a) $(2, 2, 2) = 2\mathbf{u} + 2\mathbf{v}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(b) $(3, 1, 5) = 4\mathbf{u} + 3\mathbf{v}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) $(0, 4, 5)$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(d) $(0, 0, 0) = 0\mathbf{u} + 0\mathbf{v}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

9. Similar to the process in Exercise 7, determining whether the given matrices are linear combinations of A, B, and C can be accomplished by reducing the matrix

$$\left[\begin{array}{rrr|r|r|r|r} 4 & 1 & 0 & 6 & 0 & 6 & -1 \\ 0 & -1 & 2 & -8 & 0 & 0 & 5 \\ -2 & 2 & 1 & -1 & 0 & 3 & 7 \\ -2 & 3 & 4 & -8 & 0 & 8 & 1 \end{array}\right].$$

The matrix reduces to $\left[\begin{array}{ccc|c|c|c|c} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$.

(a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = 1A + 2B - 3C$ is a linear combination of *A*, *B*, and *C*.

(b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0A + 0B + 0C$ is a linear combination of *A*, *B*, and *C*.

(c) $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix} = 1A + 2B + 1C$ is a linear combination of *A*, *B*, and *C*.

(d) $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$ is not a linear combination of *A*, *B*, and *C*.

11. Let $\mathbf{b} = (b_1, b_2, b_3)$ be an arbitrary vector in R^3.

(a) If $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{b}$, then $(2k_1, 2k_1 + k_3, 2k_1 + 3k_2 + k_3) = (b_1, b_2, b_3)$ or $\begin{aligned} 2k_1 &= b_1 \\ 2k_1 + k_3 &= b_2 \\ 2k_1 + 3k_2 + k_3 &= b_3 \end{aligned}$.

Since $\det\begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix} = -6 \neq 0$, the system is consistent for all values of b_1, b_2, and b_3 so the given vectors span R^3.

(b) If $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{b}$, then $(2k_1 + 4k_2 + 8k_3, -k_1 + k_2 - k_3, 3k_1 + 2k_2 + 8k_3) = (b_1, b_2, b_3)$ or

$$\begin{aligned} 2k_1 + 4k_2 + 8k_3 &= b_1 \\ -k_1 + k_2 - k_3 &= b_2 \\ 3k_1 + 2k_2 + 8k_3 &= b_3 \end{aligned}.$$

Since $\det\begin{bmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{bmatrix} = 0$, the system is not consistent for all values of b_1, b_2, and b_3 so the given vectors do not span R^3.

(c) If $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{b}$, then
$(3k_1 + 2k_2 + 5k_3 + k_4, k_1 - 3k_2 - 2k_3 + 4k_4, 4k_1 + 5k_2 + 9k_3 - k_4) = (b_1, b_2, b_3)$ or

$$\begin{aligned} 3k_1 + 2k_2 + 5k_3 + k_4 &= b_1 \\ k_1 - 3k_2 - 2k_3 + 4k_4 &= b_2 \\ 4k_1 + 5k_2 + 9k_3 - k_4 &= b_3 \end{aligned}.$$

Reducing the matrix $\begin{bmatrix} 3 & 2 & 5 & 1 & b_1 \\ 1 & -3 & -2 & 4 & b_2 \\ 4 & 5 & 9 & -1 & b_3 \end{bmatrix}$ leads to $\begin{bmatrix} 1 & -3 & -2 & 4 & b_2 \\ 0 & 1 & 1 & -1 & \frac{1}{11}b_1 - \frac{3}{11}b_2 \\ 0 & 0 & 0 & 0 & -\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 \end{bmatrix}$.

The system has a solution only if $-\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 = 0$ or $17b_1 = 7b_2 + 11b_3$. This restriction means that the span of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ is not all of R^3, i.e., the given vectors do not span R^3.

(d) If $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{b}$, then
$(k_1 + 3k_2 + 4k_3 + 3k_4,\ 2k_1 + 4k_2 + 3k_3 + 3k_4,\ 6k_1 + k_2 + k_3 + k_4) = (b_1, b_2, b_3)$ or

$$\begin{aligned} k_1 + 3k_2 + 4k_3 + 3k_4 &= b_1 \\ 2k_1 + 4k_2 + 3k_3 + 3k_4 &= b_2 \\ 6k_1 + k_2 + k_3 + k_4 &= b_3 \end{aligned}.$$

The matrix $\begin{bmatrix} 1 & 3 & 4 & 3 & b_1 \\ 2 & 4 & 3 & 3 & b_2 \\ 6 & 1 & 1 & 1 & b_3 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{39} & -\frac{1}{39}b_1 - \frac{1}{39}b_2 + \frac{7}{39}b_3 \\ 0 & 1 & 0 & \frac{16}{39} & -\frac{16}{39}b_1 + \frac{23}{39}b_2 - \frac{5}{39}b_3 \\ 0 & 0 & 1 & \frac{17}{39} & \frac{22}{39}b_1 - \frac{17}{39}b_2 + \frac{2}{39}b_3 \end{bmatrix}$.

Thus, for any values of $b_1, b_2,$ and b_3, values of k_1, k_2, k_3, and k_4 can be found. The given vectors span R^3.

13. Let $\mathbf{p} = a_0 + a_1x + a_2x^2$ be an arbitrary polynomial in P_2. If $k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 + k_4\mathbf{p}_4 = \mathbf{p}$, then
$(k_1 + 3k_2 + 5k_3 - 2k_4) + (-k_1 + k_2 - k_3 - 2k_4)x + (2k_1 + 4k_3 + 2k_4)x^2 = a_0 + a_1x + a_2x^2$ or

$$\begin{aligned} k_1 + 3k_2 + 5k_3 - 2k_4 &= a_0 \\ -k_1 + k_2 - k_3 - 2k_4 &= a_1 \\ 2k_1 + 4k_3 + 2k_4 &= a_2 \end{aligned}.$$

Reducing the matrix $\begin{bmatrix} 1 & 3 & 5 & -2 & a_0 \\ -1 & 1 & -1 & -2 & a_1 \\ 2 & 0 & 4 & 2 & a_2 \end{bmatrix}$ leads to $\begin{bmatrix} 1 & 3 & 5 & -2 & a_0 \\ 0 & 1 & 1 & -1 & \frac{1}{4}a_0 + \frac{1}{4}a_1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \end{bmatrix}$.

The system has a solution only if $-\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 = 0$ or $a_0 = 3a_1 + 2a_2$. This restriction means that the span of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ is not all of P_3, i.e., they do not span P_3.

15. (a) Since $\det(A) = 0$, reduce the matrix $\begin{bmatrix} -1 & 1 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & -4 & -5 & 0 \end{bmatrix}$. The matrix reduces to $\begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The solution set is $x = -\frac{1}{2}t$, $y = -\frac{3}{2}t$, $z = t$ which is a line through the origin.

(b) Since $\det(A) = 0$, reduce the matrix $\begin{bmatrix} 1 & -2 & 3 & 0 \\ -3 & 6 & 9 & 0 \\ -2 & 4 & -6 & 0 \end{bmatrix}$. The matrix reduces to $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The solution set is $x = 2t$, $y = t$, $z = 0$, which is a line through the origin.

(c) Since $\det(A) = -1 \neq 0$, the system has only the trivial solution, i.e., the origin.

(d) Since $\det(A) = 8 \neq 0$, the system has only the trivial solution, i.e., the origin.

(e) Since $\det(A) = 0$, reduce the matrix $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -1 & 4 & 0 \\ 3 & 1 & 11 & 0 \end{bmatrix}$. The matrix reduces to $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The solution set is $x = -3t$, $y = -2t$, $z = t$, which is a line through the origin.

(f) Since det(A) = 0, reduce the matrix $\begin{bmatrix} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{bmatrix}$. The matrix reduces to $\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The solution set is $x - 3y + z = 0$, which is a plane through the origin.

17. Let $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ be elements of the set. Then

$$\begin{aligned} \mathbf{f} + \mathbf{g} &= \int_a^b (f(x) + g(x))dx \\ &= \int_a^b f(x)dx + \int_a^b g(x)dx \\ &= 0 \end{aligned}$$

and $\mathbf{f} + \mathbf{g}$ is in the set. Let k be any scalar. Then $k\mathbf{f} = \int_a^b kf(x)dx = k\int_a^b f(x)dx = k \cdot 0 = 0$ so $k\mathbf{f}$ is in the set. Thus the set is a subspace of $C[a, b]$.

True/False 4.2

(a) True; this is part of the definition of a subspace.

(b) True; since a vector space is a subset of itself.

(c) False; for instance, the set W in Example 4 contains the origin (zero vector), but is not a subspace of R^2.

(d) False; R^2 is not a subset of R^3.

(e) False; if $\mathbf{b} \neq \mathbf{0}$, i.e., the system is not homogeneous, then the solution set does not contain the zero vector.

(f) True; this is by the definition of the span of a set of vectors.

(g) True; by Theorem 4.2.2.

(h) False; consider the subspaces $W_1 = \{(x, y) | y = x\}$ and $W_2 = \{(x, y) | y = -x\}$ in R^2. $\mathbf{v}_1 = (1, 1)$ is in W_1 and $\mathbf{v}_2 = (1, -1)$ is in W_2, but $\mathbf{v}_1 + \mathbf{v}_2 = (2, 0)$ is not in $W_1 \cup W_2 = \{(x, y) | y = \pm x\}$.

(i) False; let $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (0, 1)$, $\mathbf{u}_1 = (-1, 0)$, and $\mathbf{u}_2 = (0, -1)$. Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{u}_1, \mathbf{u}_2\}$ both span R^2 but $\{\mathbf{u}_1, \mathbf{u}_2\} \neq \{\mathbf{v}_1, \mathbf{v}_2\}$.

(j) True; the sum of two upper triangular matrices is upper triangular and the scalar multiple of an upper triangular matrix is upper triangular.

(k) False; the span of $x - 1$, $(x-1)^2$, and $(x-1)^3$ will only contain polynomials for which $p(1) = 0$, i.e., not all of P_3.

Section 4.3

Exercise Set 4.3

1. (a) $\mathbf{u}_2 = -5\mathbf{u}_1$, i.e., $\mathbf{u}_2$ is a scalar multiple of $\mathbf{u}_1$.

(b) Since there are 3 vectors and 3 > 2, the set is linearly dependent by Theorem 4.3.3.

(c) $\mathbf{p}_2 = 2\mathbf{p}_1$, i.e., $\mathbf{p}_2$ is a scalar multiple of $\mathbf{p}_1$.

(d) $B = -A$, i.e., B is a scalar multiple of A.

3. **(a)** The equation $k_1(3, 8, 7, -3) + k_2(1, 5, 3, -1) + k_3(2, -1, 2, 6) + k_4(1, 4, 0, 3) = (0, 0, 0, 0)$ generates the homogeneous system.

$$\begin{aligned} 3k_1 + k_2 + 2k_3 + k_4 &= 0 \\ 8k_1 + 5k_2 - k_3 + 4k_4 &= 0 \\ 7k_1 + 3k_2 + 2k_3 &= 0 \\ -3k_1 - k_2 + 6k_3 + 3k_4 &= 0 \end{aligned}$$

Since $\det\begin{bmatrix} 3 & 1 & 2 & 1 \\ 8 & 5 & -1 & 4 \\ 7 & 3 & 2 & 0 \\ -3 & -1 & 6 & 3 \end{bmatrix} = 128 \neq 0$, the system has only the trivial solution and the vectors are linearly independent.

(b) The equation $k_1(0, 0, 2, 2) + k_2(3, 3, 0, 0) + k_3(1, 1, 0, -1) = (0, 0, 0, 0)$ generates the homogeneous system

$$\begin{aligned} 3k_2 + k_3 &= 0 \\ 3k_2 + k_3 &= 0 \\ 2k_1 &= 0 \\ 2k_1 - k_3 &= 0 \end{aligned}$$

The third equation gives $k_1 = 0$, which gives $k_3 = 0$ in the fourth equation. $k_3 = 0$ gives $k_2 = 0$ in the first two equations. Since the system has only the trivial solution, the vectors are linearly independent.

(c) The equation $k_1(0, 3, -3, -6) + k_2(-2, 0, 0, -6) + k_3(0, -4, -2, -2) + k_4(0, -8, 4, -4) = (0, 0, 0, 0)$ generates the homogeneous system

$$\begin{aligned} -2k_2 &= 0 \\ 3k_1 - 4k_3 - 8k_4 &= 0 \\ -3k_1 - 2k_3 + 4k_4 &= 0 \\ -6k_1 - 6k_2 - 2k_3 - 4k_4 &= 0 \end{aligned}.$$

Since $\det\begin{bmatrix} 0 & -2 & 0 & 0 \\ 3 & 0 & -4 & -8 \\ -3 & 0 & -2 & 4 \\ -6 & -6 & -2 & -4 \end{bmatrix} = 480 \neq 0$, the system has only the trivial solution and the vectors are linearly independent.

(d) The equation $k_1(3, 0, -3, 6) + k_2(0, 2, 3, 1) + k_3(0, -2, -2, 0) + k_4(-2, 1, 2, 1) = (0, 0, 0, 0)$ generates the homogeneous system

$$\begin{aligned} 3k_1 - 2k_4 &= 0 \\ 2k_2 - 2k_3 + k_4 &= 0 \\ -3k_1 + 3k_2 - 2k_3 + 2k_4 &= 0 \\ 6k_1 + k_2 + k_4 &= 0 \end{aligned}.$$

Since $\det\begin{bmatrix} 3 & 0 & 0 & -2 \\ 0 & 2 & -2 & 1 \\ -3 & 3 & -2 & 2 \\ 6 & 1 & 0 & 1 \end{bmatrix} = 36 \neq 0$, the system has only the trivial solution and the vectors are linearly independent.

5. If the vectors lie in a plane, then they are linearly dependent.

(a) The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ generates the homogeneous system

$$\begin{aligned} 2k_1 + 6k_2 + 2k_3 &= 0 \\ -2k_1 + k_2 \qquad &= 0. \\ 4k_2 - 4k_3 &= 0 \end{aligned}$$

Since $\det\begin{bmatrix} 2 & 6 & 2 \\ -2 & 1 & 0 \\ 0 & 4 & -4 \end{bmatrix} = -72 \neq 0$, the vectors are linearly independent, so they do not lie in a plane.

(b) The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ generates the homogeneous system

$$\begin{aligned} -6k_1 + 3k_2 + 4k_3 &= 0 \\ 7k_1 + 2k_2 - k_3 &= 0. \\ 2k_1 + 4k_2 + 2k_3 &= 0 \end{aligned}$$

Since $\det\begin{bmatrix} -6 & 3 & 4 \\ 7 & 2 & -1 \\ 2 & 4 & 2 \end{bmatrix} = 0$, the vectors are linearly dependent, so they do lie in a plane.

7. (a) The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ generates the homogeneous system

$$\begin{aligned} 6k_2 + 4k_3 &= 0 \\ 3k_1 \qquad - 7k_3 &= 0 \\ k_1 + 5k_2 + k_3 &= 0. \\ -k_1 + k_2 + 3k_3 &= 0 \end{aligned}$$

The matrix $\begin{bmatrix} 0 & 6 & 4 & 0 \\ 3 & 0 & -7 & 0 \\ 1 & 5 & 1 & 0 \\ -1 & 1 & 3 & 0 \end{bmatrix}$ reduces to

$\begin{bmatrix} 1 & 0 & -\frac{7}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since the system has a nontrivial solution, the vectors are linearly dependent.

(b) The solution of the system is $k_1 = \frac{7}{3}t$, $k_2 = -\frac{2}{3}t$, $k_3 = t$.

$t = \frac{3}{7}$: $\mathbf{v}_1 = \frac{2}{7}\mathbf{v}_2 - \frac{3}{7}\mathbf{v}_3$

$t = -\frac{3}{2}$: $\mathbf{v}_2 = \frac{7}{2}\mathbf{v}_1 + \frac{3}{2}\mathbf{v}_3$

$t = 1$: $\mathbf{v}_3 = -\frac{7}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2$

9. The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ generates the homogeneous system

$$\begin{aligned} \lambda k_1 - \frac{1}{2}k_2 - \frac{1}{2}k_3 &= 0 \\ -\frac{1}{2}k_1 + \lambda k_2 - \frac{1}{2}k_3 &= 0. \\ -\frac{1}{2}k_1 - \frac{1}{2}k_2 + \lambda k_3 &= 0 \end{aligned}$$

$$\det\begin{bmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{bmatrix} = \frac{1}{4}(4\lambda^3 - 3\lambda - 1) = \frac{1}{4}(\lambda - 1)(2\lambda + 1)^2$$

For $\lambda = 1$ and $\lambda = -\frac{1}{2}$, the determinant is zero and the vectors form a linearly dependent set.

11. Let $\{\mathbf{v}_a, \mathbf{v}_b, \ldots, \mathbf{v}_n\}$ be a (nonempty) subset of *S*. If this set were linearly dependent, then there would be a nonzero solution $(k_a, k_b, \ldots, k_n)$ to $k_a\mathbf{v}_a + k_b\mathbf{v}_b + \cdots + k_n\mathbf{v}_n = \mathbf{0}$. This can be expanded to a nonzero solution of $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$ by taking all other coefficients as 0. This contradicts the linear independence of *S*, so the subset must be linearly independent.

13. If *S* is linearly dependent, then there is a nonzero solution $(k_1, k_2, \ldots, k_r)$ to $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$. Thus $(k_1, k_2, \ldots, k_r, 0, 0, \ldots, 0)$ is a nonzero solution to

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r + k_{r+1}\mathbf{v}_{r+1} + \cdots + k_n\mathbf{v}_n = \mathbf{0}$$

so the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is linearly dependent.

15. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ were linearly dependent, there would be a nonzero solution to $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ with $k_3 \neq 0$ (since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent). Solving for

$\mathbf{v}_3$ gives $\mathbf{v}_3 = -\frac{k_1}{k_3}\mathbf{v}_1 - \frac{k_2}{k_3}\mathbf{v}_2$ which contradicts that $\mathbf{v}_3$ is not in span $\{\mathbf{v}_1, \mathbf{v}_2\}$. Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

19. **(a)** If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are placed with their initial points at the origin, they will not lie in the same plane, so they are linearly independent.

(b) If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are placed with their initial points at the origin, they will lie in the same plane, so they are linearly dependent. Alternatively, note that $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_3$ by placing $\mathbf{v}_1$ along the *z*-axis with its terminal point at the origin.

21. The Wronskian is
$W(x) = \begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix} = -x\sin x - \cos x.$
$W(0) = -\cos 0 = -1$, so the functions are linearly independent.

23. **(a)** $W(x) = \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix} = e^x$

Since $W(x) = e^x$ is nonzero for all *x*, the vectors are linearly independent.

(b) $W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$

Since $W(x) = 2$ is nonzero for all *x*, the vectors are linearly independent.

25.
$$\begin{aligned} W(x) &= \begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ -\sin x & -\cos x & -x\cos x - 2\sin x \end{vmatrix} \\ &= \begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ 0 & 0 & -2\sin x \end{vmatrix} \\ &= -2\sin x \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} \\ &= -2\sin x(-\sin^2 x - \cos^2 x) \\ &= 2\sin x \end{aligned}$$
Since $2\sin x$ is not identically zero, the vectors are linearly independent, hence span a three-dimensional subspace of $F(-\infty, \infty)$. Note that the determinant was simplified by adding the first row to the third row.

True/False 4.3

(a) False; the set $\{\mathbf{0}\}$ is linearly dependent.

(b) True

(c) False; the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ in R^2 where $\mathbf{v}_1 = (0, 1)$ and $\mathbf{v}_2 = (0, 2)$ is linearly dependent.

(d) True; suppose $\{k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3\}$ were linearly dependent, then there would be a nontrivial solution to $k_1(k\mathbf{v}_1) + k_2(k\mathbf{v}_2) + k_3(k\mathbf{v}_3) = \mathbf{0}$ which would give a nontrivial solution to $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$.

(e) True; consider the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, ..., $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent, then $\mathbf{v}_2 = k\mathbf{v}_1$ for some *k* and the condition is met. If $\mathbf{v}_2 \neq k\mathbf{v}_1$ then there is some $2 < k \leq n$ for which the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}$ is linearly independent but $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent. This means that there is a nontrivial solution of $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{k-1}\mathbf{v}_{k-1} + a_k\mathbf{v}_k = \mathbf{0}$ with $a_k \neq 0$ and the equation can be solved to give $\mathbf{v}_k$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_{k-1}$.

(f) False; the set is
$\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right\}$ and
$$1\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
so the set is linearly dependent.

(g) True; the first polynomial has a nonzero constant term, so it cannot be expressed as a linear combination of the others, and since the two others are not scalar multiples of one another, the three polynomials are linearly independent.

(h) False; the functions f_1 and f_2 are linearly dependent if there are scalars k_1 and k_2 such that $k_1f_1(x)+k_2f_2(x)=0$ for all real numbers x.

Section 4.4

Exercise Set 4.4

1. (a) The set $S=\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is not linearly independent; $\mathbf{u}_3=2\mathbf{u}_1+\mathbf{u}_2$.

(b) The set $S=\{\mathbf{u}_1, \mathbf{u}_2\}$ spans a plane in R^3, not all of R^3.

(c) The set $S=\{\mathbf{p}_1, \mathbf{p}_2\}$ does not span P_2; x^2 is not a linear combination of $\mathbf{p}_1$ and $\mathbf{p}_2$.

(d) The set $S = \{A, B, C, D, E\}$ is not linearly independent; E can be written as linear combination of A, B, C, and D.

3. (a) As in Example 3, it is sufficient to consider $\det\begin{bmatrix}1&2&3\\0&2&3\\0&0&3\end{bmatrix}=6\neq 0.$

Since this determinant is nonzero, the set of vectors is a basis for R^3.

(b) It is sufficient to consider $\det\begin{bmatrix}3&2&1\\1&5&4\\-4&6&8\end{bmatrix}=26\neq 0.$ Since this determinant is nonzero, the set of vectors is a basis for R^3.

(c) It is sufficient to consider $\det\begin{bmatrix}2&4&0\\-3&1&-7\\1&1&1\end{bmatrix}=0.$ Since this determinant is zero, the set of vectors is not linearly independent.

(d) It is sufficient to consider $\det\begin{bmatrix}1&2&-1\\6&4&2\\4&-1&5\end{bmatrix}=0.$ Since this determinant is zero, the set of vectors is not linearly independent.

5. The equations $c_1\begin{bmatrix}3&6\\3&-6\end{bmatrix}+c_2\begin{bmatrix}0&-1\\-1&0\end{bmatrix}+c_3\begin{bmatrix}0&-8\\-12&-4\end{bmatrix}+c_4\begin{bmatrix}1&0\\-1&2\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ and $c_1\begin{bmatrix}3&6\\3&-6\end{bmatrix}+c_2\begin{bmatrix}0&-1\\-1&0\end{bmatrix}+c_3\begin{bmatrix}0&-8\\-12&-4\end{bmatrix}+c_4\begin{bmatrix}1&0\\-1&2\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ generate the systems

$$\begin{array}{rcl}3c_1 \qquad\qquad\quad + c_4 = 0\\ 6c_1 - c_2 \;\; - 8c_3 \qquad\quad = 0\\ 3c_1 - c_2 - 12c_3 \;\; - c_4 = 0\\ -6c_1 \qquad - 4c_3 + 2c_4 = 0\end{array} \quad\text{and}\quad \begin{array}{rcl}3c_1 \qquad\qquad\quad + c_4 = a\\ 6c_1 - c_2 \;\; - 8c_3 \qquad\quad = b\\ 3c_1 - c_2 - 12c_3 \;\; - c_4 = c\\ -6c_1 \qquad - 4c_3 + 2c_4 = d\end{array}$$

which have the same coefficient matrix. Since $\det\begin{bmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{bmatrix} = 48 \neq 0$ the matrices are linearly independent and also span M_{22}, so they are a basis.

7. **(a)** Since S is the standard basis for R^2, $(\mathbf{w})_S = (3, -7)$.

(b) Solve $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{w}$, or in terms of components, $(2c_1 + 3c_2, -4c_1 + 8c_2) = (1, 1)$. Equating components gives $\begin{aligned} 2c_1 + 3c_2 &= 1 \\ -4c_1 + 8c_2 &= 1 \end{aligned}$.

The matrix $\begin{bmatrix} 2 & 3 & 1 \\ -4 & 8 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{bmatrix}$. Thus, $(\mathbf{w})_S = \left(\frac{5}{28}, \frac{3}{14}\right)$.

(c) Solve $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{w}$, or in terms of components, $(c_1, c_1 + 2c_2) = (a, b)$. It is clear that $c_1 = a$ and solving $a + 2c_2 = b$ for c_2 gives $c_2 = \frac{b-a}{2}$. Thus, $(\mathbf{w})_S = \left(a, \frac{b-a}{2}\right)$.

9. **(a)** Solve $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$, or in terms of components, $(c_1 + 2c_2 + 3c_3, 2c_2 + 3c_3, 3c_3) = (2, -1, 3)$. Equating components gives

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 2 \\ 2c_2 + 3c_3 &= -1 \\ 3c_3 &= 3 \end{aligned}$$

which can easily be solved by back-substitution to get $c_1 = 3$, $c_2 = -2$, $c_3 = 1$. Thus, $(\mathbf{v})_S = (3, -2, 1)$.

(b) Solve $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$, or in terms of components, $(c_1 - 4c_2 + 7c_3, 2c_1 + 5c_2 - 8c_3, 3c_1 + 6c_2 + 9c_3)$ $= (5, -12, 3)$.
Equating components gives

$$\begin{aligned} c_1 - 4c_2 + 7c_3 &= 5 \\ 2c_1 + 5c_2 - 8c_3 &= -12 \\ 3c_1 + 6c_2 + 9c_3 &= 3 \end{aligned}$$

. The matrix $\begin{bmatrix} 1 & -4 & 7 & 5 \\ 2 & 5 & -8 & -12 \\ 3 & 6 & 9 & 3 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

The solution of the system is $c_1 = -2$, $c_2 = 0$, $c_3 = 1$ and $(\mathbf{v})_S = (-2, 0, 1)$.

11. Solve $c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = A$. By inspection, $c_3 = -1$ and $c_4 = 3$. Thus it remains to solve the system $\begin{aligned} -c_1 + c_2 &= 2 \\ c_1 + c_2 &= 0 \end{aligned}$ which can be solved by adding the equations to get $c_2 = 1$, from which $c_1 = -1$. Thus, $(A)_S = (-1, 1, -1, 3)$.

13. Solving $c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

$c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and

$c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = A$ gives the systems

$$\begin{aligned} c_1 \quad\quad\quad &= 0 \\ c_1 + c_2 \quad\quad &= 0 \\ c_1 + c_2 + c_3 \quad &= 0 \\ c_1 + c_2 + c_3 + c_4 &= 0 \end{aligned}, \quad \begin{aligned} c_1 \quad\quad\quad &= a \\ c_1 + c_2 \quad\quad &= b \\ c_1 + c_2 + c_3 \quad &= c \\ c_1 + c_2 + c_3 + c_4 &= d \end{aligned}, \text{ and}$$

$$\begin{aligned} c_1 \quad\quad\quad &= 1 \\ c_1 + c_2 \quad\quad &= 0 \\ c_1 + c_2 + c_3 \quad &= 1 \\ c_1 + c_2 + c_3 + c_4 &= 0 \end{aligned}.$$

It is easy to see that $\det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 1 \neq 0$, so $\{A_1, A_2, A_3, A_4\}$ spans M_{22}. The third system is easy to solve by inspection to get $c_1 = 1$, $c_2 = -1$, $c_3 = 1$, $c_4 = -1$, thus, $A = A_1 - A_2 + A_3 - A_4$.

15. Solving the systems $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$, $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = a + bx + cx^2$, and $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$ gives the systems

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 &= 0, \\ c_1+c_2+c_3 &= 0 \end{aligned} \quad \begin{aligned} c_1 &= a \\ c_1 + c_2 &= b, \text{ and} \\ c_1+c_2+c_3 &= c \end{aligned}$$

$$\begin{aligned} c_1 &= 7 \\ c_1 + c_2 &= -1. \\ c_1+c_2+c_3 &= 2 \end{aligned}$$

It is easy to see that $\det\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 1 \neq 0$, so $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ spans P_2. The third system is easy to solve by inspection to get $c_1 = 7$, $c_2 = -8$, $c_3 = 3$, thus, $\mathbf{p} = 7\mathbf{p}_1 - 8\mathbf{p}_2 + 3\mathbf{p}_3$.

17. From the diagram, $\mathbf{j} = \mathbf{u}_2$ and

$$\mathbf{u}_1 = (\cos 30°)\mathbf{i} + (\sin 30°)\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}.$$

Solving $\mathbf{u}_1 = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{u}_2$ for $\mathbf{i}$ in terms of $\mathbf{u}_1$ and $\mathbf{u}_2$ gives $\mathbf{i} = \frac{2}{\sqrt{3}}\mathbf{u}_1 - \frac{1}{\sqrt{3}}\mathbf{u}_2$.

(a) $(\sqrt{3}, 1)$ is $\sqrt{3}\mathbf{i} + \mathbf{j}$.

$$\begin{aligned} \sqrt{3}\mathbf{i} + \mathbf{j} &= \sqrt{3}\left(\frac{2}{\sqrt{3}}\mathbf{u}_1 - \frac{1}{\sqrt{3}}\mathbf{u}_2\right) + \mathbf{u}_2 \\ &= 2\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_2 \\ &= 2\mathbf{u}_1 \end{aligned}$$

The $x'y'$-coordinates are (2, 0).

(b) (1, 0) is $\mathbf{i}$ which has $x'y'$-coordinates $\left(\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

(c) Since $(0, 1) = \mathbf{j} = \mathbf{u}_2$, the $x'y'$-coordinates are (0, 1).

(d) (a, b) is $a\mathbf{i} + b\mathbf{j}$.

$$\begin{aligned} a\mathbf{i} + b\mathbf{j} &= a\left(\frac{2}{\sqrt{3}}\mathbf{u}_1 - \frac{1}{\sqrt{3}}\mathbf{u}_2\right) + b\mathbf{u}_2 \\ &= \frac{2}{\sqrt{3}}a\mathbf{u}_1 + \left(b - \frac{a}{\sqrt{3}}\right)\mathbf{u}_2 \end{aligned}$$

The $x'y'$-coordinates are $\left(\frac{2}{\sqrt{3}}a, b - \frac{a}{\sqrt{3}}\right)$.

True/False 4.4

(a) False; $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ must also be linearly independent to be a basis.

(b) False; the span of the set must also be V for the set to be a basis.

(c) True; a basis must span the vector space.

(d) True; the standard basis is used for the coordinate vectors in R^n.

(e) False; the set $\{1 + x + x^2 + x^3 + x^4,\ x + x^2 + x^3 + x^4,\ x^2 + x^3 + x^4,\ x^3 + x^4,\ x^4\}$ is a basis for P_4.

Section 4.5

Exercise Set 4.5

1. The matrix $\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The solution of the system is $x_1 = t$, $x_2 = 0$, $x_3 = t$, which can be written as $(x_1, x_2, x_3) = (t, 0, t)$ or $(x_1, x_2, x_3) = t(1, 0, 1)$. Thus, the solution space has dimension 1 and a basis is (1, 0, 1).

3. The matrix $\begin{bmatrix} 1 & -4 & 3 & -1 & 0 \\ 2 & -8 & 6 & -2 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The solution of the system is $x_1 = 4r - 3s + t$, $x_2 = r$, $x_3 = s$, $x_4 = t$, which can be written as $(x_1, x_2, x_3, x_4) = (4r - 3s + t, r, s, t)$ or

$$\begin{aligned} &(x_1, x_2, x_3, x_4) \\ &= r(4, 1, 0, 0) + s(-3, 0, 1, 0) + t(1, 0, 0, 1). \end{aligned}$$

Thus, the solution space has dimension 3 and a basis is (4, 1, 0, 0), (−3, 0, 1, 0), (1, 0, 0, 1).

5. The matrix $\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, so the system has only the trivial solution, which has dimension 0 and no basis.

7. (a) Solving the equation for x gives $x = \frac{2}{3}y - \frac{5}{3}z$, so parametric equations are $x = \frac{2}{3}r - \frac{5}{3}s$, $y = r$, $z = s$, which can be written in vector form as

$$(x, y, z) = \left(\frac{2}{3}r - \frac{5}{3}s, r, s\right) = r\left(\frac{2}{3}, 1, 0\right) + s\left(-\frac{5}{3}, 0, 1\right).$$

A basis is $\left(\frac{2}{3}, 1, 0\right)$, $\left(-\frac{5}{3}, 0, 1\right)$.

(b) Solving the equation for x gives $x = y$, so parametric equations are $x = r$, $y = r$, $z = s$, which can be written in vector form as $(x, y, z) = (r, r, s) = r(1, 1, 0) + s(0, 0, 1)$. A basis is (1, 1, 0), (0, 0, 1).

(c) In vector form, the line is $(x, y, z) = (2t, -t, 4t) = t(2, -1, 4)$. A basis is (2, –1, 4).

(d) The vectors can be parametrized as $a = r$, $b = r + s$, $c = s$ or $(a, b, c) = (r, r + s, s) = r(1, 1, 0) + s(0, 1, 1)$. A basis is (1, 1, 0), (0, 1, 1).

9. (a) The dimension is n, since a basis is the set $\{A_1, A_2, \ldots, A_n\}$ where the only nonzero entry in A_i is $a_{ii} = 1$.

(b) In a symmetric matrix, the elements above the main diagonal determine the elements below the main diagonal, so the entries on and above the main diagonal determine all the entries. There are

$$n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}$$

entries on or above the main diagonal, so the space has dimension $\frac{n(n+1)}{2}$.

(c) As in part (b), there are $\frac{n(n+1)}{2}$ entries on or above the main diagonal, so the space has dimension $\frac{n(n+1)}{2}$.

11. (a) Let $\mathbf{p}_1 = p_1(x)$ and $\mathbf{p}_2 = p_2(x)$ be elements of W. Then $(p_1 + p_2)(1) = p_1(1) + p_2(1) = 0$ so $\mathbf{p}_1 + \mathbf{p}_2$ is in W. $(kp_1)(1) = k(p_1(1)) = k \cdot 0 = 0$ so $k\mathbf{p}_1$ is in W. Thus, W is a subspace of P_2.

(b), (c) A basis for $\mathbf{W}$ is $\{-1 + x, -x + x^2\}$ so the dimension is 2.

13. Let $\mathbf{u}_1 = (1, 0, 0, 0)$, $\mathbf{u}_2 = (0, 1, 0, 0)$, $\mathbf{u}_3 = (0, 0, 1, 0)$, $\mathbf{u}_4 = (0, 0, 0, 1)$. Then the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ clearly spans R^4. The equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + k_3\mathbf{u}_3 + k_4\mathbf{u}_4 = \mathbf{0}$$

leads to the system

$$\begin{aligned} c_1 - 3c_2 + k_1 \qquad\qquad &= 0 \\ -4c_1 + 8c_2 \quad + k_2 \qquad &= 0 \\ 2c_1 - 4c_2 \qquad + k_3 \quad &= 0 \\ -3c_1 + 6c_2 \qquad\qquad + k_4 &= 0 \end{aligned}.$$

The matrix $\begin{bmatrix} 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ -4 & 8 & 0 & 1 & 0 & 0 & 0 \\ 2 & -4 & 0 & 0 & 1 & 0 & 0 \\ -3 & 6 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -2 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 \end{bmatrix}$.

From the reduced matrix it is clear that $\mathbf{u}_1$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ so it will not be in the basis, and that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent so it is one possible basis. Similarly, $\mathbf{u}_4$ and either $\mathbf{u}_2$ or $\mathbf{u}_3$ will produce a linearly independent set. Thus, any two of the vectors (0, 1, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1) can be used.

15. Let $\mathbf{v}_3 = (a, b, c)$. The equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ leads to the system

$$\begin{aligned} c_1 \qquad + ac_3 &= 0 \\ -2c_1 + 5c_2 + bc_3 &= 0. \\ 3c_1 - 3c_2 + cc_3 &= 0 \end{aligned}$$

The matrix $\begin{bmatrix} 1 & 0 & a & 0 \\ -2 & 5 & b & 0 \\ 3 & -3 & c & 0 \end{bmatrix}$ reduces to

$\begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & \frac{2}{5}a + \frac{1}{5}b & 0 \\ 0 & 0 & -\frac{9}{5}a + \frac{3}{5}b + c & 0 \end{bmatrix}$ and the homogeneous system has only the trivial solution if $-\frac{9}{5}a + \frac{3}{5}b + c \neq 0$. Thus, adding any vector $\mathbf{v}_3 = (a, b, c)$ with $9a - 3b - 5c \neq 0$ will create a basis for R^3.

True/False 4.5

(a) True; by definition.

(b) True; any basis of R^{17} will contain 17 linearly independent vectors.

(c) False; to span R^{17}, at least 17 vectors are required.

(d) True; since the dimension for R^5 is 5, then linear independence is sufficient for 5 vectors to be a basis.

(e) True; since the dimension of R^5 is 5, then spanning is sufficient for 5 vectors to be a basis.

(f) True; if the set contains n vectors, it is a basis, while if it contains more than n vectors, it can be reduced to a basis.

(g) True; if the set contains n vectors, it is a basis, while if it contains fewer than n vectors, it can be enlarged to a basis.

(h) True; the set

$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$ form a basis for M_{22}.

(i) True; the set of $n \times n$ matrices has dimension n^2, while the set $\{I, A, A^2, \ldots, A^{n^2}\}$ contains $n^2 + 1$ matrices.

(j) False; since P_2 has dimension three, then by Theorem 4.5.6(c), the only three-dimensional subspace of P_2 is P_2 itself.

Section 4.6

Exercise Set 4.6

1. (a) Since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is the standard basis B for R^2, $[\mathbf{w}]_s = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$.

(b) $\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ -4 & 8 & 0 & 1 \end{array}\right]$ reduces to

$\left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{7} & -\frac{3}{28} \\ 0 & 1 & \frac{1}{7} & \frac{1}{14} \end{array}\right]$. Thus $P_{B\to S} = \begin{bmatrix} \frac{2}{7} & -\frac{3}{28} \\ \frac{1}{7} & \frac{1}{14} \end{bmatrix}$

and

$$[\mathbf{w}]_S = P_{B\to S}[\mathbf{w}]_B = \begin{bmatrix} \frac{2}{7} & -\frac{3}{28} \\ \frac{1}{7} & \frac{1}{14} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{28} \\ \frac{3}{14} \end{bmatrix}.$$

(c) $\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$ reduces to $\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$.

Thus $P_{B\to S} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and

$$\begin{aligned} [\mathbf{w}]_S &= P_{B\to S}[\mathbf{w}]_B \\ &= \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} a \\ -\frac{1}{2}a + \frac{1}{2}b \end{bmatrix}. \end{aligned}$$

3. (a) Since S is the standard basis B for P_2,

$(\mathbf{p})_S = (4, -3, 1)$ or $[\mathbf{p}]_S = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$.

(b) $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ reduces to

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$. Thus,

$P_{B\to S} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and

$$\begin{aligned}[\mathbf{p}]_S &= P_{B\to S}[\mathbf{p}]_B \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}\end{aligned}$$

or $(\mathbf{p})_S = (0, 2, -1)$.

5. (a) From Theorem 4.6.2, $P_{S\to B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$ where B is the standard basis for R^3. Thus,

$$[\mathbf{w}]_B = P_{S\to B}[\mathbf{w}]_S = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \\ 12 \end{bmatrix}$$

and **w** = (16, 10, 12).

(b) The basis in Exercise 3(*a*) is the standard basis B for P_2, so $[\mathbf{q}]_B = [\mathbf{q}]_S = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$ and $\mathbf{q} = 3 + 4x^2$.

(c) From Theorem 4.6.2,

$P_{S\to E} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ where E is the standard basis for M_{22}.

Thus,

$$\begin{aligned}[B]_E = P_{S\to E}[B]_S &= \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -8 \\ 7 \\ 6 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 15 \\ -1 \\ 6 \\ 3 \end{bmatrix}\end{aligned}$$

and $B = \begin{bmatrix} 15 & -1 \\ 6 & 3 \end{bmatrix}$.

7. (a) $\left[\begin{array}{cc|cc} 2 & 4 & 1 & -1 \\ 2 & -1 & 3 & -1 \end{array}\right]$ reduces to

$\left[\begin{array}{cc|cc} 1 & 0 & \frac{13}{10} & -\frac{1}{2} \\ 0 & 1 & -\frac{2}{5} & 0 \end{array}\right]$ so $P_{B'\to B} = \begin{bmatrix} \frac{13}{10} & -\frac{1}{2} \\ -\frac{2}{5} & 0 \end{bmatrix}$.

(b) $\left[\begin{array}{cc|cc} 1 & -1 & 2 & 4 \\ 3 & -1 & 2 & -1 \end{array}\right]$ reduces to

$\left[\begin{array}{cc|cc} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & -2 & -\frac{13}{2} \end{array}\right]$ so $P_{B\to B'} = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}$.

(c) $\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right]$ reduces to

$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{10} & \frac{2}{5} \\ 0 & 1 & \frac{1}{5} & -\frac{1}{5} \end{array}\right]$ so $P_{E\to B} = \begin{bmatrix} \frac{1}{10} & \frac{2}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$

where E is the standard basis for R^2.

$$\begin{aligned}[\mathbf{w}]_B &= P_{E\to B}[\mathbf{w}]_E \\ &= \begin{bmatrix} \frac{1}{10} & \frac{2}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix}\end{aligned}$$

$$\begin{aligned}[\mathbf{w}]_{B'} &= P_{B\to B'}[\mathbf{w}]_B \\ &= \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}\begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -7 \end{bmatrix}\end{aligned}$$

(d) $\left[\begin{array}{rr|rr} 1 & -1 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right]$ reduces to $\left[\begin{array}{rr|rr} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$

so $P_{E\to B'} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$.

$$\begin{aligned}[\mathbf{w}]_{B'} &= P_{E\to B'}[\mathbf{w}]_E \\ &= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -7 \end{bmatrix}\end{aligned}$$

9. (a) $\left[\begin{array}{rrr|rrr} 3 & 1 & -1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array}\right]$ reduces to

$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & 2 & \frac{5}{2} \\ 0 & 1 & 0 & -2 & -3 & -\frac{1}{2} \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right]$ so

$P_{B\to B'} = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix}$.

(b) $\left[\begin{array}{rrr|rrr} 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ reduces to

$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right]$ so

$P_{E\to B} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -1 & 0 & 2 \end{bmatrix}$ where E is the standard basis for R^3.

$$\begin{aligned}[\mathbf{w}]_B &= P_{E\to B}[\mathbf{w}]_E \\ &= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}[\mathbf{w}]_{B'} &= P_{B\to B'}[\mathbf{w}]_B \\ &= \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix}\begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{bmatrix}\end{aligned}$$

(c) $\left[\begin{array}{rrr|rrr} 3 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -5 & -3 & 2 & 0 & 0 & 1 \end{array}\right]$ reduces to

$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right]$ so

$P_{E\to B'} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 2 & 1 \end{bmatrix}$.

$$\begin{aligned}[\mathbf{w}]_{B'} &= P_{E\to B'}[\mathbf{w}]_E \\ &= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{bmatrix}\end{aligned}$$

11. (a) The span of $\mathbf{f}_1$ and $\mathbf{f}_2$ is the set of all linear combinations $a\mathbf{f}_1 + b\mathbf{f}_2 = a\sin x + b\cos x$ and this vector can be represented by (a, b). Since $\mathbf{g}_1 = 2\mathbf{f}_1 + \mathbf{f}_2$ and $\mathbf{g}_2 = 3\mathbf{f}_2$, it is sufficient to compute $\det\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = 6$. Since this determinant is nonzero, $\mathbf{g}_1$ and $\mathbf{g}_2$ form a basis for V.

(b) Since B can be represented as {(1, 0), (0, 1)} $P_{B'\to B} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$.

(c) $\left[\begin{array}{rr|rr} 2 & 0 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$ reduces to $\left[\begin{array}{rr|rr} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{6} & \frac{1}{3} \end{array}\right]$

so $P_{B\to B'} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$.

(d) $[\mathbf{h}]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ since B is the standard basis for V.

$$[\mathbf{h}]_{B'} = P_{B\to B'}[\mathbf{h}]_B = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

13. (a) $P_{B\to S} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

(b) $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$ reduces to

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$ so

$$P_{S\to B} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

(d)
$$\begin{aligned}[\mathbf{w}]_B &= P_{S\to B}[\mathbf{w}]_S \\ &= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -239 \\ 77 \\ 30 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}[\mathbf{w}]_S &= P_{B\to S}[\mathbf{w}]_B \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}\begin{bmatrix} -239 \\ 77 \\ 30 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}\end{aligned}$$

(e) $[\mathbf{w}]_S = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$

$$\begin{aligned}[\mathbf{w}]_B &= P_{S\to B}[\mathbf{w}]_S \\ &= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -200 \\ 64 \\ 25 \end{bmatrix}\end{aligned}$$

15. (a) $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 4 \end{array}\right]$ reduces to $\left[\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & -1 & -2 \end{array}\right]$

so $P_{B_2\to B_1} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$.

(b) $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 3 & 4 & 2 & 3 \end{array}\right]$ reduces to $\left[\begin{array}{cc|cc} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \end{array}\right]$

so $P_{B_1\to B_2} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$.

(d) $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right]$ reduces to $\left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right]$

so $P_{S\to B_1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$ where S is the standard basis $\{(1, 0), (0, 1)\}$ for R^2. Since $\mathbf{w}$ is the standard basis vector (0, 1) for R^2,

$[\mathbf{w}]_{B_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$$\begin{aligned}[\mathbf{w}]_{B_2} &= P_{B_1\to B_2}[\mathbf{w}]_{B_1} \\ &= \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}\end{aligned}$$

(e) $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$ reduces to $\left[\begin{array}{cc|cc} 1 & 0 & 4 & -1 \\ 0 & 1 & -3 & 1 \end{array}\right]$

so $P_{S\to B_2} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$.

$$[\mathbf{w}]_{B_2} = P_{S\to B_2}[\mathbf{w}]_S = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{aligned}[\mathbf{w}]_{B_1} &= P_{B_2\to B_1}[\mathbf{w}]_{B_2} \\ &= \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ -1 \end{bmatrix}\end{aligned}$$

17. (a) $\left[\begin{array}{rrr|rrr} 3 & 1 & -1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array}\right]$ reduces to

$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & 2 & \frac{5}{2} \\ 0 & 1 & 0 & -2 & -3 & -\frac{1}{2} \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right]$ so

$$P_{B_1 \to B_2} = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix}.$$

(b) $\left[\begin{array}{rrr|rrr} 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ reduces to

$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right]$, so

$P_{S \to B_1} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -1 & 0 & 2 \end{bmatrix}$ where S is the standard basis for R^3.

$$\begin{aligned} [\mathbf{w}]_{B_1} &= P_{S \to B_1}[\mathbf{w}]_S \\ &= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} [\mathbf{w}]_{B_2} &= P_{B_1 \to B_2}[\mathbf{w}]_{B_1} \\ &= \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix}\begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{bmatrix} \end{aligned}$$

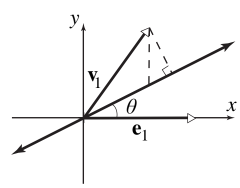
(c) $\left[\begin{array}{rrr|rrr} 3 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -5 & -3 & 2 & 0 & 0 & 1 \end{array}\right]$ reduces to

$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right]$ so

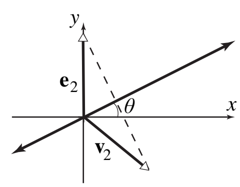
$$P_{S \to B_2} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 2 & 1 \end{bmatrix}.$$

$$\begin{aligned} [\mathbf{w}]_{B_2} &= P_{S \to B_2}[\mathbf{w}]_S \\ &= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{bmatrix} \end{aligned}$$

19. (a)



From the diagram, it is clear that $[\mathbf{v}_1]_S = (\cos 2\theta, \sin 2\theta)$.



From the diagram, the angle between the positive y-axis and $\mathbf{v}_2$ is $2\left(\frac{\pi}{2} - \theta\right) = \pi - 2\theta$, so the angle between $\mathbf{v}_2$ and the positive x-axis is $-\left(\pi - 2\theta - \frac{\pi}{2}\right) = 2\theta - \frac{\pi}{2}$ and

$$\begin{aligned} [\mathbf{v}_2]_S &= \left(\cos\left(2\theta - \frac{\pi}{2}\right), \sin\left(2\theta - \frac{\pi}{2}\right)\right) \\ &= (\sin 2\theta, -\cos 2\theta) \end{aligned}$$

Thus by Theorem 4.6.2,

$$P_{B \to S} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

21. If **w** is a vector in the space and $[\mathbf{w}]_{B'}$ is the coordinate vector of **w** relative to B', then $[\mathbf{w}]_B = P[\mathbf{w}]_{B'}$ and $[\mathbf{w}]_C = Q[\mathbf{w}]_B$, so $[\mathbf{w}]_C = Q(P[\mathbf{w}]_{B'}) = QP[\mathbf{w}]_{B'}$. The transition matrix from B' to C is QP. The transition matrix from C to B' is $(QP)^{-1} = P^{-1}Q^{-1}$.

23. **(a)** By Theorem 4.6.2, P is the transition matrix from $B = \{(1, 1, 0), (1, 0, 2), (0, 2, 1)\}$ to S.

(b) $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$ reduces to

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{4}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{array}\right].$$

Thus, P is the transition matrix from S to

$$B = \left\{\left(\frac{4}{5}, \frac{1}{5}, -\frac{2}{5}\right), \left(\frac{1}{5}, -\frac{1}{5}, \frac{2}{5}\right), \left(-\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)\right\}$$

27. B must be the standard basis for R^n, since, in particular $[\mathbf{e}_i]_B = \mathbf{e}_i$ for the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

True/False 4.6

(a) True

(b) True; by Theorem 4.6.1.

(c) True

(d) True

(e) False; consider the bases $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $B = \{2\mathbf{e}_3, 3\mathbf{e}_1, 5\mathbf{e}_2\}$ for R^3, then $P_{B\to S} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 5 \\ 2 & 0 & 0 \end{bmatrix}$ which is not a diagonal matrix.

(f) False; A would have to be invertible, not just square.

Section 4.7

Exercise Set 4.7

1. Row vectors: $\mathbf{r}_1 = [2 \ \ -1 \ \ 0 \ \ 1]$, $\mathbf{r}_2 = [3 \ \ 5 \ \ 7 \ \ -1]$, $\mathbf{r}_3 = [1 \ \ 4 \ \ 2 \ \ 7]$

Column vectors: $\mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$, $\mathbf{c}_3 = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$, $\mathbf{c}_4 = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$

3. **(a)** $\begin{bmatrix} 1 & 3 & -2 \\ 4 & -6 & 10 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$, so $\begin{bmatrix} -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ so the system $A\mathbf{x} = \mathbf{b}$ is inconsistent and $\mathbf{b}$ is not in the column space of A.

(c) $\begin{bmatrix} 1 & -1 & 1 & 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, so $\begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the system $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{b}$ has the solution $x_1 = 1$, $x_2 = t - 1$, $x_3 = t$.

Thus $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + (t-1)\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(e) $\begin{bmatrix} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 1 & 2 & 1 & 3 & 5 \\ 0 & 1 & 2 & 2 & 7 \end{bmatrix}$ reduces to

$\begin{bmatrix} 1 & 0 & 0 & 0 & -26 \\ 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$, so

$$\begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix} = -26\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 13\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} - 7\begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}.$$

5. (a) $\begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and the solution of $A\mathbf{x} = \mathbf{b}$ is $x_1 = 1 + 3t$, $x_2 = t$, or in vector form, $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

The general form of the solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 0 & 1 & -2 \\ 2 & 1 & 3 & 3 \end{bmatrix}$ reduces to

$\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and the solution of $A\mathbf{x} = \mathbf{b}$ is $x_1 = -2 - t$, $x_2 = 7 - t$, $x_3 = t$, or in vector form, $\mathbf{x} = \begin{bmatrix} -2 \\ 7 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. The general form of the solution of $A\mathbf{x} = \mathbf{0}$ is

$\mathbf{x} = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & -2 & 1 & 2 & -1 \\ 2 & -4 & 2 & 4 & -2 \\ -1 & 2 & -1 & -2 & 1 \\ 3 & -6 & 3 & 6 & -3 \end{bmatrix}$ reduces to

$\begin{bmatrix} 1 & -2 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and the solution of

$A\mathbf{x} = \mathbf{b}$ is $x_1 = -1 + 2r - s - 2t$, $x_2 = r$, $x_3 = s$, $x_4 = t$, or in vector form

$$\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The general form of the solution of $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(d) $\begin{bmatrix} 1 & 2 & -3 & 1 & 4 \\ -2 & 1 & 2 & 1 & -1 \\ -1 & 3 & -1 & 2 & 3 \\ 4 & -7 & 0 & -5 & -5 \end{bmatrix}$ reduces to

$\begin{bmatrix} 1 & 0 & -\frac{7}{5} & -\frac{1}{5} & \frac{6}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and the solution of

$A\mathbf{x} = \mathbf{b}$ is $x_1 = \frac{6}{5} + \frac{7}{5}s + \frac{1}{5}t$,

$x_2 = \frac{7}{5} + \frac{4}{5}s - \frac{3}{5}t$, $x_3 = s$, $x_4 = t$, or in

vector form, $\mathbf{x} = \begin{bmatrix} \frac{6}{5} \\ \frac{7}{5} \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{bmatrix}$.

The general form of the solution of $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = s\begin{bmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{bmatrix}.$$

7. (a) A basis for the row space is $\mathbf{r}_1 = [1 \quad 0 \quad 2]$, $\mathbf{r}_2 = [0 \quad 0 \quad 1]$. A basis for the column space is $\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

(b) A basis for the row space is $\mathbf{r}_1 = [1 \quad -3 \quad 0 \quad 0]$, $\mathbf{r}_2 = [0 \quad 1 \quad 0 \quad 0]$.

A basis for the column space is $\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(c) A basis for the row space is $\mathbf{r}_1 = [1 \quad 2 \quad 4 \quad 5]$, $\mathbf{r}_2 = [0 \quad 1 \quad -3 \quad 0]$, $\mathbf{r}_3 = [0 \quad 0 \quad 1 \quad -3]$, $\mathbf{r}_4 = [0 \quad 0 \quad 0 \quad 1]$.

A basis for the column space is $\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$\mathbf{c}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}_3 = \begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}_4 = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$.

(d) A basis for the row space is $\mathbf{r}_1 = [1 \quad 2 \quad -1 \quad 5]$, $\mathbf{r}_2 = [0 \quad 1 \quad 4 \quad 3]$, $\mathbf{r}_3 = [0 \quad 0 \quad 1 \quad -7]$, $\mathbf{r}_4 = [0 \quad 0 \quad 0 \quad 1]$.

A basis for the column space is $\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$\mathbf{c}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}_3 = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{c}_4 = \begin{bmatrix} 5 \\ 3 \\ -7 \\ 1 \end{bmatrix}$.

9. (a) A basis for the row space is $\mathbf{r}_1 = [1 \quad 0 \quad 2]$, $\mathbf{r}_2 = [0 \quad 0 \quad 1]$. A basis for the column space is $\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

(b) A basis for the row space is $\mathbf{r}_1 = [1 \quad -3 \quad 0 \quad 0]$, $\mathbf{r}_2 = [0 \quad 1 \quad 0 \quad 0]$.

A basis for the column space is $\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(c) A basis for the row space is $\mathbf{r}_1 = [1 \quad 2 \quad 4 \quad 5]$, $\mathbf{r}_2 = [0 \quad 1 \quad -3 \quad 0]$, $\mathbf{r}_3 = [0 \quad 0 \quad 1 \quad -3]$, $\mathbf{r}_4 = [0 \quad 0 \quad 0 \quad 1]$.

A basis for the column space is $\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$\mathbf{c}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}_3 = \begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}_4 = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$.

(d) A basis for the row space is $\mathbf{r}_1 = [1 \quad 2 \quad -1 \quad 5]$, $\mathbf{r}_2 = [0 \quad 1 \quad 4 \quad 3]$, $\mathbf{r}_3 = [0 \quad 0 \quad 1 \quad -7]$, $\mathbf{r}_4 = [0 \quad 0 \quad 0 \quad 1]$.

A basis for the column space is $\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$\mathbf{c}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}_3 = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{c}_4 = \begin{bmatrix} 5 \\ 3 \\ -7 \\ 1 \end{bmatrix}$.

11. (a) A row echelon form of $\begin{bmatrix} 1 & 1 & -4 & -3 \\ 2 & 0 & 2 & -2 \\ 2 & -1 & 3 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$. Thus a basis for the subspace is $(1, 1, -4, -3)$, $(0, 1, -5, -2)$, $\left(0, 0, 1, -\dfrac{1}{2}\right)$.

(b) A row echelon form of $\begin{bmatrix} -1 & 1 & -2 & 0 \\ 3 & 3 & 6 & 0 \\ 9 & 0 & 0 & 3 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{6} \end{bmatrix}$. Thus a basis for the subspace is (1, −1, 2, 0), (0, 1, 0, 0), $\left(0, 0, 1, -\frac{1}{6}\right)$.

(c) A row echelon form of $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & 0 & 2 & 2 \\ 0 & -3 & 0 & 3 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Thus a basis for the subspace is (1, 1, 0, 0), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1).

15. (a) The row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so the general solution of the system $A\mathbf{x} = \mathbf{0}$ is $x = 0$, $y = 0$, $z = t$ or $t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Thus the null space of A is the z-axis, and the column space is the span of $\mathbf{c}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, which is all linear combinations of y and x, i.e., the xy-plane.

(b) Reversing the reasoning in part (a), it is clear that one such matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

17. (a) If the null space of a matrix A is the given line, then the equation $A\mathbf{x} = \mathbf{0}$ is equivalent to $\begin{bmatrix} 3 & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and a general form for A is $\begin{bmatrix} 3a & -5a \\ 3b & -5b \end{bmatrix}$ where a and b are real numbers, not both of which are zero.

(b) $\det(A) = \det(B) = 5$
A and B are invertible so the systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have only the trivial solution. That is, the null spaces of A and B are both the origin.
The second row of C is a multiple of the first and it is clear that if $3x + y = 0$, then (x, y) is in the null space of C, i.e., the null space of C is the line $3x + y = 0$.
The equation $D\mathbf{x} = \mathbf{0}$ has all $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ as solutions, so the null space of D is the entire xy-plane.

True/False 4.7

(a) True; by the definition of column space.

(b) False; the system $A\mathbf{x} = \mathbf{b}$ may not be consistent.

(c) False; the column vectors of A that correspond to the column vectors of R containing the leading 1's form a basis for the column space of A.

(d) False; the row vectors of A may be linearly dependent if A is not in row echelon form.

(e) False; the matrices $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ have the same row space but different column spaces.

(f) True; elementary row operations do not change the null space of a matrix.

(g) True; elementary row operations do not change the row space of a matrix.

(h) False; see (e) for an example.

(i) True; this is the contrapositive of Theorem 4.7.1.

(j) False; assuming that A and B are both $n \times n$ matrices, the row space of A will have n vectors in its basis since being invertible means that A is row equivalent to I_n. However, since B is singular, it is not row equivalent to I_n, so in reduced row echelon form it has at least one row of zeros. Thus the row space of B will have fewer than n vectors in its basis.

Section 4.8

Exercise Set 4.8

1. The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -\frac{6}{7} & -\frac{4}{7} \\ 0 & 1 & \frac{17}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so rank $(A) = 2$.

$A^T = \begin{bmatrix} 1 & -3 & -2 \\ 2 & 1 & 3 \\ 4 & 5 & 9 \\ 0 & 2 & 2 \end{bmatrix}$ and the reduced row echelon form of A^T is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so

$\text{rank}(A^T) = \text{rank}(A) = 2.$

3. **(a)** Since rank $(A) = 2$, there are 2 leading variables. Since nullity$(A) = 3 - 2 = 1$, there is 1 parameter.

(b) Since rank$(A) = 1$, there is 1 leading variable. Since nullity$(A) = 3 - 1 = 2$, there are 2 parameters.

(c) Since rank$(A) = 2$, there are 2 leading variables. Since nullity$(A) = 4 - 2 = 2$, there are 2 parameters.

(d) Since rank$(A) = 2$, there are 2 leading variables. Since nullity$(A) = 5 - 2 = 3$, there are 3 parameters.

(e) Since rank$(A) = 3$, there are 3 leading variables. Since nullity$(A) = 5 - 3 = 2$, there are 2 parameters.

5. **(a)** Since A is 4×4, rank(A) + nullity$(A) = 4$, and the largest possible value of rank(A) is 4, so the smallest possible value of nullity(A) is 0.

(b) Since A is 3×5, rank(A) + nullity$(A) = 5$, and the largest possible value of rank(A) is 3, so the smallest possible value of nullity(A) is 2.

(c) Since A is 5×3, rank(A) + nullity$(A) = 3$, and the largest possible value of rank(A) is 3, so the smallest possible value of nullity(A) is 0.

7. **(a)** Since $\text{rank}(A) = \text{rank}[A|\mathbf{b}]$, the system is consistent, and since nullity$(A) = 3 - 3 = 0$, there are 0 parameters in the general solution.

(b) Since $\text{rank}(A) < \text{rank}[A|\mathbf{b}]$, the system is inconsistent.

(c) Since $\text{rank}(A) = \text{rank}[A|\mathbf{b}]$, the system is consistent, and since nullity$(A) = 3 - 1 = 2$, there are 2 parameters in the general solution.

(d) Since $\text{rank}(A) = \text{rank}[A|\mathbf{b}]$, the system is consistent, and since nullity$(A) = 9 - 2 = 7$, there are 7 parameters in the general solution.

(e) Since $\text{rank}(A) < \text{rank}[A|\mathbf{b}]$, the system is inconsistent.

(f) Since $\text{rank}(A = \text{rank}[A|\mathbf{b}]$, the system is consistent, and since nullity$(A) = 4 - 0 = 4$, there are 4 parameters in the general solution.

(g) Since $\text{rank}(A) = \text{rank}[A|\mathbf{b}]$, the system is consistent, and since nullity$(A) = 2 - 2 = 0$, there are 0 parameters in the general solution.

9. The augmented matrix is $\begin{bmatrix} 1 & -3 & b_1 \\ 1 & -2 & b_2 \\ 1 & 1 & b_3 \\ 1 & -4 & b_4 \\ 1 & 5 & b_5 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 0 & 3b_2 - 2b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 - 4b_2 + 3b_1 \\ 0 & 0 & b_4 + b_2 - 2b_1 \\ 0 & 0 & b_5 - 8b_2 + 7b_1 \end{bmatrix}$. Thus, the system is consistent if and only if

$$\begin{aligned} 3b_1 - 4b_2 + b_3 \qquad\qquad &= 0 \\ -2b_1 + b_2 \qquad + b_4 \qquad &= 0 \\ 7b_1 - 8b_2 \qquad\qquad + b_5 &= 0 \end{aligned}$$

which has the general solution $b_1 = r,\ b_2 = s,$ $b_3 = -3r + 4s,\ b_4 = 2r - s,\ b_5 = -7r + 8s.$

11. No, since rank(A) + nullity(A) = 3 and nullity(A) = 1, the row space and column space must both be 2-dimensional, i.e., planes in 3-space.

13. The rank is 2 if $r = 2$ and $s = 1$. Since the third column will always have a nonzero entry, the rank will never be 1.

17. **(a)** The number of leading 1's is at most 3, since a matrix cannot have more leading 1's than rows.

(b) The number of parameters is at most 5, since the solution cannot have more parameters than A has columns.

(c) The number of leading 1's is at most 3, since the row space of A has the same dimension as the column space, which is at most 3.

(d) The number of parameters is at most 3 since the solution cannot have more parameters than A has columns.

19. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then rank($A$) = rank($B$) = 1, but since $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B^2 = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$,

$\text{rank}(A^2) = 0 \neq \text{rank}(B^2) = 1$.

True/False 4.8

(a) False; the row vectors are not necessarily linearly independent, and if they are not, then neither are the column vectors, since dim(row space) = dim(column space).

(b) True; if the row vectors are linearly independent then nullity(A) = 0 and rank(A) = n = the number of rows. But, since rank(A) + nullity(A) = the number of columns, A must be square.

(c) False; the nullity of a nonzero $m \times n$ matrix is at most $n - 1$.

(d) False; if the added column is linearly dependent on the previous columns, adding it does not affect the rank.

(e) True; if the rows are linearly dependent, then the rank is at least 1 less than the number of rows, so since the matrix is square, its nullity is at least 1.

(f) False; if the nullity of A is zero, then $A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$.

(g) False; the dimension of the row space is always equal to the dimension of the column space.

(h) False; $\text{rank}(A) = \text{rank}(A^T)$ for all matrices.

(i) True; the row space and null space cannot both have dimension 1 since dim(row space) + dim(null space) = 3.

(j) False; a vector $\mathbf{w}$ in $W^\perp$ need not be orthogonal to every vector in V; the relationship is that $V^\perp$ is a subspace of $W^\perp$.

Section 4.9

Exercise Set 4.9

1. **(a)** Since A has size 3×2, the domain of T_A is R^2 and the codomain of T_A is R^3.

(b) Since A has size 2×3, the domain of T_A is R^3 and the codomain of T_A is R^2.

(c) Since A has size 3×3, the domain of T_A is R^3 and the codomain of T_A is R^3.

(d) Since A has size 1×6, the domain of T_A is R^6 and the codomain of T_A is R^1.

3. The domain of T is R^2 and the codomain of T is R^3.
$T(1, -2) = (1 - 2, -(-2), 3(1)) = (-1, 2, 3)$

5. **(a)** The transformation is linear, with domain R^3 and codomain R^2.

(b) The transformation is nonlinear, with domain R^2 and codomain R^3.

(c) The transformation is linear, with domain R^3 and codomain R^3.

(d) The transformation is nonlinear, with domain R^4 and codomain R^2.

7. (a) T is a matrix transformation;
$T_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) T is not a matrix transformation since $T(0, 0, 0) \neq \mathbf{0}$.

(c) T is a matrix transformation;
$T_A = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -5 \end{bmatrix}$.

(d) T is not a matrix transformation, since for $k \neq 1$,
$$\begin{aligned} T(k(x, y, z)) &= (k^2y^2, kz) \\ &\neq kT(x, y, z) \\ &= (ky^2, kz). \end{aligned}$$

(e) T is not a matrix transformation, since $T(0, 0, 0) = (-1, 0) \neq \mathbf{0}$.

9. By inspection, $T = \begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}$.

For $(x, y, z) = (-1, 2, 4)$,
$w_1 = 3(-1) + 5(2) - 4 = 3$
$w_2 = 4(-1) - (2) + 4 = -2$
$w_3 = 3(-1) + 2(2) - (4) = -3$
so $T(-1, 2, 4) = (3, -2, -3)$.
Also, $\begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$.

11. (a) The matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{bmatrix}$.

(b) The matrix is $\begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$.

(c) The matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(d) The matrix is $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$.

13. (a) $A_T\mathbf{x} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$
$T(-1, 4) = (-(-1) + 4, 4) = (5, 4)$

(b) $A_T\mathbf{x} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$
$$\begin{aligned} T(2, 1, -3) &= (2(2) - (1) + (-3), 1 + (-3), 0) \\ &= (0, -2, 0) \end{aligned}$$

15. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$
The result is $(2, -5, -3)$.

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$
The result is $(2, 5, 3)$.

(c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$
The result is $(-2, -5, 3)$.

17. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$
The result is $(-2, 1, 0)$.

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$
The result is $(-2, 0, 3)$.

(c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$
The result is $(0, 1, 3)$.

19. (a) $\begin{bmatrix}1 & 0 & 0\\ 0 & \cos 30° & -\sin 30°\\ 0 & \sin 30° & \cos 30°\end{bmatrix}\begin{bmatrix}-2\\ 1\\ 2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2}\\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}-2\\ 1\\ 2\end{bmatrix}$

$=\begin{bmatrix}-2\\ \frac{\sqrt{3}-2}{2}\\ \frac{1+2\sqrt{3}}{2}\end{bmatrix}$

The result is $\left(-2, \frac{\sqrt{3}-2}{2}, \frac{1+2\sqrt{3}}{2}\right)$.

(b) $\begin{bmatrix}\cos 45° & 0 & \sin 45°\\ 0 & 1 & 0\\ -\sin 45° & 0 & \cos 45°\end{bmatrix}\begin{bmatrix}-2\\ 1\\ 2\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-2\\ 1\\ 2\end{bmatrix}$

$=\begin{bmatrix}0\\ 1\\ 2\sqrt{2}\end{bmatrix}$

The result is $(0, 1, 2\sqrt{2})$.

(c) $\begin{bmatrix}\cos 90° & -\sin 90° & 0\\ \sin 90° & \cos 90° & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-2\\ 1\\ 2\end{bmatrix}=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-2\\ 1\\ 2\end{bmatrix}$

$=\begin{bmatrix}-1\\ -2\\ 2\end{bmatrix}$

The result is $(-1, -2, 2)$.

21. (a) $\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(-30°) & -\sin(-30°)\\ 0 & \sin(-30°) & \cos(-30°)\end{bmatrix}\begin{bmatrix}-2\\ 1\\ 2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2}\\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}-2\\ 1\\ 2\end{bmatrix}$

$=\begin{bmatrix}-2\\ \frac{\sqrt{3}+2}{2}\\ \frac{-1+2\sqrt{3}}{2}\end{bmatrix}$

The result is $\left(-2, \frac{\sqrt{3}+2}{2}, \frac{-1+2\sqrt{3}}{2}\right)$.

(b) $\begin{bmatrix} \cos(-45°) & 0 & \sin(-45°) \\ 0 & 1 & 0 \\ -\sin(-45°) & 0 & \cos(-45°) \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} -2\sqrt{2} \\ 1 \\ 0 \end{bmatrix}$

The result is $\left(-2\sqrt{2}, 1, 0\right)$.

(c) $\begin{bmatrix} \cos(-90°) & -\sin(-90°) & 0 \\ \sin(-90°) & \cos(-90°) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

The result is (1, 2, 2).

25. $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{(2, 2, 1)}{\sqrt{2^2+2^2+1^2}} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) = (a, b, c)$

$\cos 180° = -1$, $\sin 180° = 0$

The matrix is

$$\begin{bmatrix} \left(\frac{2}{3}\right)^2(1-(-1))+(-1) & \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)(1-(-1))-\frac{1}{3}(0) & \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)(1-(-1))+\left(\frac{2}{3}\right)(0) \\ \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)(1-(-1))+\frac{1}{3}(0) & \left(\frac{2}{3}\right)^2(1-(-1))+(-1) & \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)(1-(-1))-\left(\frac{2}{3}\right)(0) \\ \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)(1-(-1))-\frac{2}{3}(0) & \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)(1-(-1))+\frac{2}{3}(0) & \left(\frac{1}{3}\right)^2(1-(-1))+(-1) \end{bmatrix} = \begin{bmatrix} \frac{4}{9}(2)-1 & \frac{4}{9}(2) & \frac{2}{9}(2) \\ \frac{4}{9}(2) & \frac{4}{9}(2)-1 & \frac{2}{9}(2) \\ \frac{2}{9}(2) & \frac{2}{9}(2) & \frac{1}{9}(2)-1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{9} & \frac{8}{9} & \frac{4}{9} \\ \frac{8}{9} & -\frac{1}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{4}{9} & -\frac{7}{9} \end{bmatrix}$$

29. (a) The result is twice the orthogonal projection of **x** on the *x*-axis.

(b) The result is twice the reflection of **x** about the *x*-axis.

31. Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, the matrix can be written as $\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$ which has the effect of rotating **x** through the angle 2θ.

33. The result of the transformation is rotation through the angle θ followed by translation by $\mathbf{x}_0$. It is not a matrix transformation since $T(\mathbf{0}) = \mathbf{x}_0 \neq \mathbf{0}$.

35. The image of the line under the operator *T* is the line in R^n which has vector representation $\mathbf{x} = T(\mathbf{x}_0) + tT(\mathbf{v})$, unless *T* is the zero operator in which case the image is **0**.

True/False 4.9

(a) False; if A is 2×3, then the domain of T_A is R^3.

(b) False; if A is $m \times n$, then the codomain of T_A is R^m.

(c) False; $T(\mathbf{0}) = \mathbf{0}$ is not a sufficient condition. For instance if T: $R^n \to R^1$ is given by $T(\mathbf{v}) = \|\mathbf{v}\|$ then $T(\mathbf{0}) = \mathbf{0}$, but T is not a matrix transformation.

(d) True

(e) False; every matrix transformation will have this property by the homogeneity property.

(f) True; the only matrix transformation with this property is the zero transformation (consider $\mathbf{x} = \mathbf{y} \neq \mathbf{0}$).

(g) False; since $\mathbf{b} \neq \mathbf{0}$, then $T(\mathbf{0}) = \mathbf{0} + \mathbf{b} = \mathbf{b} \neq \mathbf{0}$, so T cannot be a matrix operator.

(h) False; there is no angle θ for which $\cos\theta = \sin\theta = \dfrac{1}{2}$.

(i) True; when $a = 1$ it is the matrix for reflection about the x-axis and when $a = -1$ it is the matrix for reflection about the y-axis.

Section 4.10

Exercise Set 4.10

1. The standard matrix for $T_B \circ T_A$ is

$$BA = \begin{bmatrix} 2 & -3 & 3 \\ 5 & 0 & 1 \\ 6 & 1 & 7 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 4 & 1 & -3 \\ 5 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 21 \\ 10 & -8 & 4 \\ 45 & 3 & 25 \end{bmatrix}.$$

The standard matrix for $T_A \circ T_B$ is

$$AB = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 1 & -3 \\ 5 & 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & -3 & 3 \\ 5 & 0 & 1 \\ 6 & 1 & 7 \end{bmatrix} = \begin{bmatrix} -8 & -3 & 1 \\ -5 & -15 & -8 \\ 44 & -11 & 45 \end{bmatrix}.$$

3. (a) The standard matrix for T_1 is $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and the standard matrix for T_2 is $\begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix}$.

(b) The standard matrix for $T_2 \circ T_1$ is

$$\begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 6 & -2 \end{bmatrix}.$$

The standard matrix for $T_1 \circ T_2$ is

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 1 & -4 \end{bmatrix}.$$

(c) $T_1(T_2(x_1, x_2)) = (5x_1 + 4x_2,\ x_1 - 4x_2)$

$T_2(T_1(x_1, x_2)) = (3x_1 + 3x_2,\ 6x_1 - 2x_2)$

5. (a) The standard matrix for a rotation of 90° is $\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and the standard matrix for reflection about the line $y = x$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(b) The standard matrix for orthogonal projection on the y-axis is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and the standard matrix for a contraction with factor $k = \dfrac{1}{2}$ is $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

(c) The standard matrix for reflection about the x-axis is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and the standard matrix for a dilation with factor $k = 3$ is $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

7. **(a)** The standard matrix for reflection about the yz-plane is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the standard matrix for orthogonal projection on the xz-plane is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) The standard matrix for a rotation of 45° about the y-axis is $\begin{bmatrix} \cos 45° & 0 & \sin 45° \\ 0 & 1 & 0 \\ -\sin 45° & 0 & \cos 45° \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ and the standard matrix for a dilation with factor $k = \sqrt{2}$ is $\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$.

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(c) The standard matrix for orthogonal projection on the xy-plane is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the standard matrix for reflection about the yz-plane is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

9. **(a)** $[T_1] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$[T_1][T_2] = [T_2][T_1] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so $T_1 \circ T_2 = T_2 \circ T_1$.

(b) From Example 1, $[T_2 \circ T_1] = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$

$$\begin{aligned}
[T_1 \circ T_2] &= [T_1][T_2] \\
&= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \\
&= [T_2 \circ T_1]
\end{aligned}$$

Thus $T_1 \circ T_2 = T_2 \circ T_1$.

(c) $[T_1] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$$\begin{aligned}
[T_1][T_2] &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
[T_2][T_1] &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix}
\end{aligned}$$

$T_1 \circ T_2 \neq T_2 \circ T_1$

11. (a) Not one-to-one because the second row (column) is all 0's, so the determinant is 0.

(b) One-to-one because the determinant is –1.

(c) One-to-one because the determinant is –1.

(d) One-to-one because the determinant is $k^2 \neq 0$.

(e) One-to-one because the determinant is 1.

(f) One-to-one because the determinant is –1.

(g) One-to-one because the determinant is $k^3 \neq 0$.

13. (a) $[T] = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

det$[T]$ = 1 + 2 = 3 ≠ 0, so T is one-to-one.

$$[T^{-1}] = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$T^{-1}(w_1, w_2) = \left(\frac{1}{3}w_1 - \frac{2}{3}w_2, \frac{1}{3}w_1 + \frac{1}{3}w_2\right)$$

(b) $[T]=\begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$

det[T] = 12 − 12 = 0, so T is not one-to-one.

(c) $[T]=\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

det[T] = 0 − 1 = −1 ≠ 0, so T is one-to-one.

$[T^{-1}]=\frac{1}{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$T^{-1}(w_1, w_2)=(-w_2, -w_1)$

(d) $[T]=\begin{bmatrix} 3 & 0 \\ -5 & 0 \end{bmatrix}$

det[T] = 3(0) − (−5)(0) = 0, so T is not one-to-one.

15. **(a)** The inverse is (another) reflection about the x-axis in R^2.

(b) The inverse is rotation through the angle $-\frac{\pi}{4}$ in R^2.

(c) The inverse is contraction by a factor of $\frac{1}{3}$ in R^2.

(d) The inverse is (another) reflection about the yz-plane in R^3.

(e) The inverse is dilation by a factor of 5 in R^3.

17. Let $\mathbf{u}=(x_1, y_1)$, $\mathbf{v}=(x_2, y_2)$, and k be a scalar.

(a)
$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(x_1+x_2,\ y_1+y_2) \\ &= (2(x_1+x_2)+(y_1+y_2),\ (x_1+x_2)-(y_1+y_2)) \\ &= ((2x_1+y_1)+(2x_2+y_2),\ (x_1-y_1)+(x_2-y_2)) \\ &= T(\mathbf{u})+T(\mathbf{v}) \end{aligned}$$
$$\begin{aligned} T(k\mathbf{u}) &= T(kx_1,\ ky_1) \\ &= (2kx_1+ky_1,\ kx_1-ky_1) \\ &= k(2x_1+y_1,\ x_1-y_1) \\ &= kT(\mathbf{u}) \end{aligned}$$

T is a matrix operator.

(b)
$$\begin{aligned} T(k\mathbf{u}) &= T(kx_1,\ ky_1) \\ &= (kx_1+1,\ ky_1) \\ &\neq k(x_1+1,\ y_1) \end{aligned}$$

T is not a matrix operator, which can also be shown by considering $T(\mathbf{u}+\mathbf{v})$.

(c) $T(\mathbf{u}+\mathbf{v}) = T(x_1 + x_2,\ y_1 + y_2)$
$= (y_1 + y_2,\ y_1 + y_2)$
$= T(\mathbf{u}) + T(\mathbf{v})$
$T(k\mathbf{u}) = T(kx_1,\ ky_1)$
$= (ky_1,\ ky_1)$
$= k(y_1,\ y_1)$
$= kT(\mathbf{u})$
T is a matrix operator.

(d) $T(\mathbf{u}+\mathbf{v}) = T(x_1 + x_2,\ y_1 + y_2)$
$= \left(\sqrt[3]{x_1 + x_2},\ \sqrt[3]{y_1 + y_2}\right)$
$\neq \left(\sqrt[3]{x_1} + \sqrt[3]{x_2},\ \sqrt[3]{y_1} + \sqrt[3]{y_2}\right)$
T is not a matrix operator, which can also be shown by considering $T(k\mathbf{u})$.

19. Let $\mathbf{u} = (x_1,\ y_1,\ z_1),\ \ \mathbf{v} = (x_2,\ y_2,\ z_2),$ and k be a scalar.

(a) $T(\mathbf{u}+\mathbf{v}) = T(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$
$= (0, 0)$
$= T(\mathbf{u}) + T(\mathbf{v})$
$T(k\mathbf{u}) = T(kx_1,\ ky_1,\ kz_1)$
$= (0, 0)$
$= k(0, 0)$
$= kT(\mathbf{u})$
T is a matrix transformation.

(b) $T(\mathbf{u}+\mathbf{v}) = T(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$
$= (3(x_1 + x_2) - 4(y_1 + y_2),\ 2(x_1 + x_2) - 5(z_1 + z_2))$
$= ((3x_1 - 4y_1) + (3x_2 - 4y_2),\ (2x_1 - 5z_1) + (2x_2 - 5z_2))$
$= T(\mathbf{u}) + T(\mathbf{v})$
$T(k\mathbf{u}) = T(kx_1,\ ky_1,\ kz_1)$
$= (3kx_1 - 4ky_1,\ 2kx_1 - 5kz_1)$
$= k(3x_1 - 4y_1,\ 2x_1 - 5z_1)$
$= kT(\mathbf{u})$
T is a matrix operator.

21. (a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$
$= \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$

23. (a) $T_A(\mathbf{e}_1) = (-1, 2, 4),\ T_A(\mathbf{e}_2) = (3, 1, 5),$ and $T_A(\mathbf{e}_3) = (0, 2, -3)$

(b) $T_A(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)$
$=(-1+3+0,\ 2+1+2,\ 4+5-3)$
$=(2, 5, 6)$

(c) $T_A(7\mathbf{e}_3)=7(0, 2, -3)=(0, 14, -21)$

25. (a) Yes; if **u** and **v** are distinct vectors in the domain of T_1 which is one-to-one, then $T_1(\mathbf{u})$ and $T_1(\mathbf{v})$ are distinct. If $T_1(\mathbf{u})$ and $T_1(\mathbf{v})$ are in the domain of T_2 which is also one-to-one, then $T_2(T_1(\mathbf{u}))$ and $T_2(T_1(\mathbf{v}))$ are distinct. Thus $T_2 \circ T_1$ is one-to-one.

(b) Yes; if $T_1(x, y)=(x, y, 0)$ and $T_2(x, y, z)=(x, y)$, then T_1 is one-to-one but T_2 is not one-to-one and the composition $T_2 \circ T_1$ is the identity transformation on R^2 which is one-to-one. However, when T_1 is not one-to-one and T_2 is one-to-one, the composition $T_2 \circ T_1$ is not one-to-one, since $T_1(\mathbf{u})=T_1(\mathbf{v})$ for some distinct vectors **u** and **v** in the domain of T_1 and consequently $T_2(T_1(\mathbf{u}))=T_2(T_1(\mathbf{v}))$ for those same vectors.

27. (a) The product of any $m\times n$ matrix and the $n\times 1$ matrix of all zeros (the zero vector in R^n) is the $m\times 1$ matrix of all zero, which is the zero vector in R^m.

(b) One example is $T(x_1, x_2)=(x_1^2+x_2^2, x_1x_2)$

29. (a) The range of T must be a proper subset of R^n, i.e., not all of R^n. Since $\det(A) = 0$, there is some $\mathbf{b}_1$ in R^n such that $A\mathbf{x}=\mathbf{b}_1$ is inconsistent, hence $\mathbf{b}_1$ is not in the range of T.

(b) T must map infinitely many vectors to **0**.

True/False 4.10

(a) False; for example T: $R^2 \to R^2$ given by $T(x_1, x_2)=(x_1^2+x_2^2, x_1x_2)$ is not a matrix transformation, although $T(\mathbf{0}) = T(0, 0) = (0, 0) = \mathbf{0}$.

(b) True; use $c_1=c_2=1$ to get the additivity property and $c_1=k,\ c_2=0$ to get the homogeneity property.

(c) True; if T is a one-to-one matrix transformation with matrix A, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. That is $T(\mathbf{x}-\mathbf{y}) = \mathbf{0} \Rightarrow A(\mathbf{x}-\mathbf{y}) = \mathbf{0}$ $\Rightarrow \mathbf{x} = \mathbf{y}$, so there are no such distinct vectors.

(d) False; the zero transformation is one example.

(e) False; the zero transformation is one example.

(f) False; the zero transformation is one example.

Section 4.11

Exercise Set 4.11

1. (a) Reflection about the line $y = -x$ maps (1, 0) to (0, −1) and (0, 1) to (−1, 0), so the standard matrix is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

(b) Reflection through the origin maps (1, 0) to (−1, 0) and (0, 1) to (0, −1), so the standard matrix is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

(c) Orthogonal projection on the x-axis maps (1, 0) to (1, 0) and (0, 1) to (0, 0), so the standard matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(d) Orthogonal projection on the y-axis maps (1, 0) to (0, 0) and (0, 1) to (0, 1), so the standard matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

3. (a) A reflection through the xy-plane maps (1, 0, 0) to (1, 0, 0), (0, 1, 0) to (0, 1, 0), and (0, 0, 1) to (0, 0, −1), so the standard matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(b) A reflection through the xz-plane maps (1, 0, 0) to (1, 0, 0), (0, 1, 0) to (0, −1, 0), and (0, 0, 1) to (0, 0, 1), so the standard matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) A reflection through the yz-plane maps (1, 0, 0) to (–1, 0, 0), (0, 1, 0) to (0, 1, 0), and (0, 0, 1) to (0, 0, 1), so the standard matrix is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

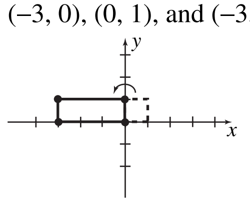
5. (a) Rotation of 90° about the z-axis maps (1, 0, 0) to (0, 1, 0), (0, 1, 0) to (–1, 0, 0), and (0, 0, 1) to (0, 0, 1), so the standard matrix is $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Rotation of 90° about the x-axis maps (1, 0, 0) to (1, 0, 0), (0, 1, 0) to (0, 0, 1), and (0, 0, 1) to (0, –1, 0), so the standard matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

(c) Rotation of 90° about the y-axis maps (1, 0, 0) to (0, 0, –1), (0, 1, 0) to (0, 1, 0), and (0, 0, 1) to (1, 0, 0), so the standard matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

7. $\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$,
$\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

The image is a rectangle with vertices at (0, 0), (–3, 0), (0, 1), and (–3, 1).



9. (a) Shear by a factor of $k = 4$ in the y-direction has matrix $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$.

(b) Shear by a factor of $k = -2$ in the x-direction has matrix $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

11. (a) The geometric effect is expansion by a factor of 3 in the x-direction.

(b) The geometric effect is expansion by a factor of 5 in the y-direction and reflection about the x-axis.

(c) The geometric effect is shearing by a factor of 4 in the x-direction.

13. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} \cos 180° & -\sin 180° \\ \sin 180° & \cos 180° \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

15. (a) Since $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, the inverse transformation is reflection about $y = x$.

(b) Since for $0 < k < 1$, $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$, the inverse transformation is an expansion along the same axis.

(c) Since $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ the inverse transformation is a reflection about the same coordinate axis.

(d) Since $k \neq 0$, $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$, the inverse transformation is a shear (in the opposite direction) along the same axis.

17. The line $y = 2x$ can be represented by $\begin{bmatrix} t \\ 2t \end{bmatrix}$.

(a) $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} t \\ 2t \end{bmatrix} = \begin{bmatrix} t+6t \\ 0+2t \end{bmatrix} = \begin{bmatrix} 7t \\ 2t \end{bmatrix}$

The image is $x = 7t$, $y = 2t$, and using $t = \frac{1}{7}x$, this is the line $y = \frac{2}{7}x$.

(b) $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} t \\ 2t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$

The image is $x = t$, $y = t$ which is the line $y = x$.

(c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} t \\ 2t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$

The image is $x = 2t$, $y = t$, which is the line $y = \frac{1}{2}x$.

(d) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} t \\ 2t \end{bmatrix} = \begin{bmatrix} -t \\ 2t \end{bmatrix}$

The image is $x = -t$, $y = 2t$, which is the line $y = -2x$.

(e) $\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix}\begin{bmatrix} t \\ 2t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} t \\ 2t \end{bmatrix}$

$$= \begin{bmatrix} \frac{1-2\sqrt{3}}{2}t \\ \frac{\sqrt{3}+2}{2}t \end{bmatrix}$$

The image is $x = \frac{1-2\sqrt{3}}{2}t$, $y = \frac{\sqrt{3}+2}{2}t$, and using $t = \frac{2}{1-2\sqrt{3}}x$, this is the line $y = \frac{\sqrt{3}+2}{1-2\sqrt{3}}x$ or $y = -\frac{8+5\sqrt{3}}{11}x$.

19. (a) $\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+y \\ 6x+2y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

Thus, the image (x', y') satisfies $y' = 2x'$ for every (x, y) in the plane.

(b) No, since the matrix is not invertible.

23. (a) To map (x, y, z) to $(x + kz, y + kz, z)$, the standard matrix is $\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Shear in the xz-direction with factor k maps (x, y, z) to $(x + ky, y, z + ky)$. The standard matrix is $\begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$.

Shear in the yz-direction with factor k maps (x, y, z) to $(x, y + kx, z + kx)$.

The standard matrix is $\begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$.

True/False 4.11

(a) False; the image will be a parallelogram, but it will not necessarily be a square.

(b) True; an invertible matrix operator can be expressed as a product of elementary matrices, each of which has geometric effect among those listed.

(c) True; by Theorem 4.11.3.

(d) True; $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(e) False; $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ maps the unit square to the parallelogram with vertices (0, 0), (1, 1), (1, −1), and (2, 0) which is not a reflection about a line.

(f) False; $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ maps the unit square to the parallelogram with vertices (0, 0), (1, 2), (−2, 1), and (−1, 3) which is not a shear in either the x- or y-direction.

(g) True; this is an expansion by a factor of 3 in the y-direction.

Section 4.12

Exercise Set 4.12

1. (a) A is a stochastic matrix.

(b) A is not a stochastic matrix since the sums of the entries in each column are not 1.

(c) A is a stochastic matrix.

(d) A is not a stochastic matrix because the third column is not a probability vector due to the negative entry.

3. $\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}$

$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} = \begin{bmatrix} 0.545 \\ 0.455 \end{bmatrix}$

$\mathbf{x}_3 = P\mathbf{x}_2 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 0.545 \\ 0.455 \end{bmatrix} = \begin{bmatrix} 0.5455 \\ 0.4545 \end{bmatrix}$

$\mathbf{x}_4 = P\mathbf{x}_3 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 0.5455 \\ 0.4545 \end{bmatrix} = \begin{bmatrix} 0.54545 \\ 0.45455 \end{bmatrix}$

$\mathbf{x}_4 = P^4\mathbf{x}_0 = \begin{bmatrix} 0.5455 & 0.5454 \\ 0.4545 & 0.4546 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

$= \begin{bmatrix} 0.54545 \\ 0.45455 \end{bmatrix}$

5. **(a)** P is a regular stochastic matrix.

(b) P is not a regular stochastic matrix, since the entry in row 1, column 2 of P^k will be 0 for all $k \geq 1$.

(c) P is a regular stochastic matrix since

$$P^2 = \begin{bmatrix} \frac{21}{25} & \frac{1}{5} \\ \frac{4}{25} & \frac{4}{5} \end{bmatrix}.$$

7. Since P is a stochastic matrix with all positive entries, P is a regular stochastic matrix.
The system $(I - P)\mathbf{q} = \mathbf{0}$ is

$$\begin{bmatrix} \frac{3}{4} & -\frac{2}{3} \\ -\frac{3}{4} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\begin{bmatrix} \frac{3}{4} & -\frac{2}{3} & 0 \\ -\frac{3}{4} & \frac{2}{3} & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & -\frac{8}{9} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so the general solution is $q_1 = \frac{8}{9}s$, $q_2 = s$. Since $\mathbf{q}$ must be a probability vector $1 = q_1 + q_2 = \frac{17}{9}s$ so $s = \frac{9}{17}$. The steady-state vector is

$$\mathbf{q} = \begin{bmatrix} \frac{8}{9}\left(\frac{9}{17}\right) \\ \frac{9}{17} \end{bmatrix} = \begin{bmatrix} \frac{8}{17} \\ \frac{9}{17} \end{bmatrix}.$$

9. Each column of P is a probability vector and

$P^2 = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{3}{8} & \frac{7}{18} \\ \frac{7}{24} & \frac{1}{8} & \frac{4}{9} \end{bmatrix}$, so P is a regular stochastic matrix. The system $(I - P)\mathbf{q} = \mathbf{0}$ is

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{4} & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The reduced row echelon form of $I - P$ is

$\begin{bmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}$, so the general solution of the system is $q_1 = \frac{4}{3}s$, $q_2 = \frac{4}{3}s$, $q_3 = s$. Since $\mathbf{q}$ must be a probability vector,

$1 = q_1 + q_2 + q_3 = \frac{11}{3}s$, so $s = \frac{3}{11}$.

The steady-state vector is $\mathbf{q} = \begin{bmatrix} \frac{4}{3}\left(\frac{3}{11}\right) \\ \frac{4}{3}\left(\frac{3}{11}\right) \\ \frac{3}{11} \end{bmatrix} = \begin{bmatrix} \frac{4}{11} \\ \frac{4}{11} \\ \frac{3}{11} \end{bmatrix}$.

11. **(a)** The entry $0.2 = p_{11}$ represents the probability that something in state 1 stays in state 1.

(b) The entry $0.1 = p_{12}$ represents the probability that something in state 2 moves to state 1.

(c) $\begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$

The probability is 0.8.

(d) $\begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.15 \\ 0.85 \end{bmatrix}$

The probability is 0.85.

13. (a) Let state 1 be good air. The transition matrix is $P=\begin{bmatrix}0.95 & 0.55\\0.05 & 0.45\end{bmatrix}$.

(b) $P^2\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0.93 & 0.77\\0.07 & 0.23\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0.93\\0.07\end{bmatrix}$

The probability is 0.93 that the air will be good.

(c) $P^3\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0.922 & 0.858\\0.078 & 0.142\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0.858\\0.142\end{bmatrix}$

The probability is 0.142 that the air will be bad.

(d) $P\begin{bmatrix}0.2\\0.8\end{bmatrix}=\begin{bmatrix}0.95 & 0.55\\0.05 & 0.45\end{bmatrix}\begin{bmatrix}0.2\\0.8\end{bmatrix}=\begin{bmatrix}0.63\\0.37\end{bmatrix}$

The probability is 0.63 that the air will be good.

15. Let state 1 be living in the city. Then the transition matrix is $P=\begin{bmatrix}0.95 & 0.03\\0.05 & 0.97\end{bmatrix}$. The initial state vector is

$\mathbf{x}_0=\begin{bmatrix}100\\25\end{bmatrix}$ (in thousands).

(a) $P\mathbf{x}_0=\begin{bmatrix}0.95 & 0.03\\0.05 & 0.97\end{bmatrix}\begin{bmatrix}100\\25\end{bmatrix}=\begin{bmatrix}95.75\\29.25\end{bmatrix}$

$P^2\mathbf{x}_0=\begin{bmatrix}0.95 & 0.03\\0.05 & 0.97\end{bmatrix}^2\begin{bmatrix}100\\25\end{bmatrix}=\begin{bmatrix}91.84\\33.16\end{bmatrix}$

$P^3\mathbf{x}_0=\begin{bmatrix}0.95 & 0.03\\0.05 & 0.97\end{bmatrix}^3\begin{bmatrix}100\\25\end{bmatrix}\approx\begin{bmatrix}88.243\\36.757\end{bmatrix}$

$P^4\mathbf{x}_0=\begin{bmatrix}0.95 & 0.03\\0.05 & 0.97\end{bmatrix}^4\begin{bmatrix}100\\25\end{bmatrix}\approx\begin{bmatrix}84.933\\40.067\end{bmatrix}$

$P^5\mathbf{x}_0=\begin{bmatrix}0.95 & 0.03\\0.05 & 0.97\end{bmatrix}^5\begin{bmatrix}100\\25\end{bmatrix}\approx\begin{bmatrix}81.889\\43.111\end{bmatrix}$

Year	**1**	**2**	**3**	**4**	**5**
City	95,750	91,840	88,243	84,933	81,889
Suburbs	29,250	33,160	36,757	40,067	43,111

(b) The system $(I-P)\mathbf{q}=\mathbf{0}$ is $\begin{bmatrix}0.05 & -0.03\\-0.05 & 0.03\end{bmatrix}\begin{bmatrix}q_1\\q_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$.

The reduced row echelon form of $I-P$ is $\begin{bmatrix}1 & -0.6\\0 & 0\end{bmatrix}$ so the general solution is $q_1=0.6s,\ \ q_2=s$. Since

$1=q_1+q_2=1.6s,\ \ s=\frac{1}{1.6}=0.625$ and the steady-state vector is $q=\begin{bmatrix}0.6(0.625)\\0.625\end{bmatrix}=\begin{bmatrix}0.375\\0.625\end{bmatrix}$.

Over the long term, the population of 125,000 will be distributed with 0.375(125,000) = 46,875 in the city and 0.625(125,000) = 78,125 in the suburbs.

17. The transition matrix is $P = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} & \frac{3}{5} \\ \frac{4}{5} & \frac{3}{10} & \frac{1}{5} \\ \frac{1}{10} & \frac{1}{2} & \frac{1}{5} \end{bmatrix}$.

(a) $P^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{23}{100} & \frac{19}{50} & \frac{11}{50} \\ \frac{17}{50} & \frac{7}{20} & \frac{29}{50} \\ \frac{43}{100} & \frac{27}{100} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{23}{100} \\ \frac{17}{50} \\ \frac{43}{100} \end{bmatrix}$

The probability is $\frac{23}{100}$.

(b) The system $(I - P)\mathbf{q} = \mathbf{0}$ is

$$\begin{bmatrix} \frac{9}{10} & -\frac{1}{5} & -\frac{3}{5} \\ -\frac{4}{5} & \frac{7}{10} & -\frac{1}{5} \\ -\frac{1}{10} & -\frac{1}{2} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The reduced row echelon form of $I - P$ is

$\begin{bmatrix} 1 & 0 & -\frac{46}{47} \\ 0 & 1 & -\frac{66}{47} \\ 0 & 0 & 0 \end{bmatrix}$, so the general solution is

$q_1 = \frac{46}{47}s,\ q_2 = \frac{66}{47}s,\ q_3 = s.$

Since $1 = q_1 + q_2 + q_3 = \frac{159}{47}s,\ s = \frac{47}{159}$ and the steady-state vector is

$$\mathbf{q} = \begin{bmatrix} \frac{46}{47}\left(\frac{47}{159}\right) \\ \frac{66}{47}\left(\frac{47}{159}\right) \\ \frac{47}{159} \end{bmatrix} = \begin{bmatrix} \frac{46}{159} \\ \frac{22}{53} \\ \frac{47}{159} \end{bmatrix}.$$

(c) $120\mathbf{q} \approx 120 \begin{bmatrix} 0.2893 \\ 0.4151 \\ 0.2956 \end{bmatrix} \approx \begin{bmatrix} 35 \\ 50 \\ 35 \end{bmatrix}$

Over the long term, the probability that a car ends up at location 1 is about 0.2893, location 2 is about 0.4151, and location 3 is about 0.2956. Thus the vector $120\mathbf{q}$ gives the long term distribution of 120 cars. The locations should have 35, 50, and 35 parking spaces, respectively.

19. Column 1: $1 - \frac{7}{10} - \frac{1}{10} = \frac{1}{5}$

Column 2: $1 - \frac{3}{10} - \frac{3}{5} = \frac{1}{10}$

Column 3: $1 - \frac{1}{5} - \frac{3}{10} = \frac{1}{2}$

$$P = \begin{bmatrix} \frac{7}{10} & \frac{1}{10} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{10} & \frac{1}{2} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \end{bmatrix}$$

The system $(I - P)\mathbf{q} = \mathbf{0}$ is

$$\begin{bmatrix} \frac{3}{10} & -\frac{1}{10} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{7}{10} & -\frac{1}{2} \\ -\frac{1}{10} & -\frac{3}{5} & \frac{7}{10} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The reduced row echelon form of $I - P$ is

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so the general solution is $q_1 = s$, $q_2 = s,\ q_3 = s.$ Since $1 = q_1 + q_2 + q_3 = 3s$,

$s = \frac{1}{3}$ and the steady-state vector is $\mathbf{q} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

21. Since $\mathbf{q}$ is the steady-state vector for P, $P\mathbf{q} = \mathbf{q}$, hence

$P^k\mathbf{q} = P^{k-1}(P\mathbf{q}) = P^{k-1}\mathbf{q} = P^{k-2}(P\mathbf{q}) = \cdots = \mathbf{q}$

and $P^k\mathbf{q} = \mathbf{q}$ for every positive integer k.

True/False 4.12

(a) True; the sum of the entries is 1 and all entries are nonnegative.

(b) True; the column vectors are probability vectors and $\begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0.84 & 0.2 \\ 0.16 & 0.8 \end{bmatrix}$.

(c) True; this is part of the definition of a transition matrix.

(d) False; the steady-state vector is a solution which is also a probability vector.

(e) True

Chapter 4 Supplementary Exercises

1. (a) $\mathbf{u} + \mathbf{v} = (3 + 1, -2 + 5, 4 + (-2)) = (4, 3, 2)$

$(-1)\mathbf{u} = ((-1)3, 0, 0) = (-3, 0, 0)$

(b) V is closed under addition because addition in V is component-wise addition of real numbers and the real numbers are closed under addition. Similarly, V is closed under scalar multiplication because the real numbers are closed under multiplication.

(c) Axioms 1–5 hold in V because they hold in R^3.

(d) Axiom 7:
$$\begin{aligned} k(\mathbf{u}+\mathbf{v}) &= k(u_1+v_1, u_2+v_2, u_3+v_3) \\ &= (k(u_1+v_1), 0, 0) \\ &= (ku_1, 0, 0)+(kv_1, 0, 0) \\ &= k\mathbf{u}+k\mathbf{v} \end{aligned}$$
Axiom 8:
$$\begin{aligned} (k+m)\mathbf{u} &= ((k+m)u_1, 0, 0) \\ &= (ku_1, 0, 0)+(mu_1, 0, 0) \\ &= k\mathbf{u}+m\mathbf{u} \end{aligned}$$
Axiom 9: $$\begin{aligned} k(m\mathbf{u}) &= k(mu_1, 0, 0) \\ &= (kmu_1, 0, 0) \\ &= (km)\mathbf{u} \end{aligned}$$

(e) For any $\mathbf{u}=(u_1, u_2, u_3)$ with $u_2, u_3 \neq 0$, $1\mathbf{u} = (1u_1, 0, 0) \neq (u_1, u_2, u_3) = \mathbf{u}$.

3. The coefficient matrix is $A = \begin{bmatrix} 1 & 1 & s \\ 1 & s & 1 \\ s & 1 & 1 \end{bmatrix}$ and $\det(A) = -(s-1)^2(s+2)$, so for $s \neq 1, -2$, A is invertible and the solution space is the origin.
If $s = 1$, then $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, which reduced to $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so the solution space will require 2 parameters, i.e., it is a plane through the origin.
If $s = -2$, then $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so the solution space will require 1 parameter, i.e., it will be a line through the origin.

7. A must be an invertible matrix for $A\mathbf{v}_1$, $A\mathbf{v}_2, \ldots, A\mathbf{v}_n$ to be linearly independent.

9. (a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so rank($A$) = 2 and nullity($A$) = 1.

(b) $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so rank($A$) = 2 and nullity($A$) = 2.

(c) The row vectors $\mathbf{r}_i$ will satisfy $\mathbf{r}_i = \begin{cases} \mathbf{r}_1 & \text{if } i \text{ is odd} \\ \mathbf{r}_2 & \text{if } i \text{ is even} \end{cases}$. Thus, the reduced row echelon form will have 2 nonzero rows. Its rank will be 2 and its nullity will be $n - 2$.

11. (a) If p_1 and p_2 are in the set, then
$$\begin{aligned} (p_1+p_2)(-x) &= p_1(-x)+p_2(-x) \\ &= p_1(x)+p_2(x) \\ &= (p_1+p_2)(x) \end{aligned}$$
so the set is closed under addition.
Also
$(kp_1)(-x) = k(p_1(-x)) = k(p_1(x)) = (kp_1)(x)$
so kp_1 is in the set for any scalar k, and the set is a subspace of P_n, the subspace consisting of all even polynomials. A basis is $\{1, x^2, x^4, \ldots, x^{2m}\}$ where $2m = n$ if n is even and $2m = n - 1$ if n is odd.

(b) If p_1 and p_2 are in the set, then $(p_1+p_2)(0) = p_1(0)+p_2(0) = 0$, so p_1+p_2 is in the set. Also, $(kp_1)(0) = k(p_1(0)) = k(0) = 0$, so kp_1 is in the set. Thus, the set is a subspace of P_n, the subspace consisting of polynomials with constant term 0. A basis is $\{x, x^2, x^3, \ldots, x^n\}$.

13. **(a)** The entry a_{ij}, $i > j$, below the main diagonal is determined by the a_{ji} entry above the main diagonal. A basis is $\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$

(b) In a skew-symmetric matrix, $a_{ij} = \begin{cases}0 \text{ if } i = j\\ -a_{ji} \text{ if } i \neq j\end{cases}$. Thus, the entries below the main diagonal are determined by the entries above the main diagonal. A basis is $\left\{\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}\right\}$.

15. Every 3×3, 4×4, and 5×5 submatrix will have determinant 0, so the rank can be at most 2. For $i < 5$ and $j < 6$, $\det\begin{bmatrix}0 & a_{i6}\\ a_{5j} & a_{56}\end{bmatrix} = -a_{i6}a_{5j}$ so the rank is 2 if any of these products is nonzero, i.e., if the matrix has 2 nonzero entries other than a_{56}. If exactly one entry in the last column is nonzero, the matrix will have rank 1, while if all entries are zero, the rank will be 0. Thus, the possible ranks are 0, 1, or 2.

Chapter 5

Eigenvalues and Eigenvectors

Section 5.1

Exercise Set 5.1

1. $A\mathbf{x} = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 5\mathbf{x}$

$\mathbf{x}$ corresponds to the eigenvalue $\lambda = 5$.

3. (a) $\det\begin{bmatrix} \lambda-3 & 0 \\ -8 & \lambda+1 \end{bmatrix} = (\lambda-3)(\lambda+1)$

$= \lambda^2 - 2\lambda - 3$

The characteristic equation is $\lambda^2 - 2\lambda - 3 = 0$.

(b) $\det\begin{bmatrix} \lambda-10 & 9 \\ -4 & \lambda+2 \end{bmatrix} = (\lambda-10)(\lambda+2) + 36$

$= \lambda^2 - 8\lambda + 16$

The characteristic equation is $\lambda^2 - 8\lambda + 16 = 0$.

(c) $\det\begin{bmatrix} \lambda & -3 \\ -4 & \lambda \end{bmatrix} = \lambda^2 - 12$

The characteristic equation is $\lambda^2 - 12 = 0$.

(d) $\det\begin{bmatrix} \lambda+2 & 7 \\ -1 & \lambda-2 \end{bmatrix} = (\lambda+2)(\lambda-2) + 7$

$= \lambda^2 + 3$

The characteristic equation is $\lambda^2 + 3 = 0$.

(e) $\det\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \lambda^2$

The characteristic equation is $\lambda^2 = 0$.

(f) $\det\begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{bmatrix} = (\lambda-1)^2 = \lambda^2 - 2\lambda + 1$

The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$.

5. (a) $\begin{bmatrix} \lambda-3 & 0 \\ -8 & \lambda+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 3$ gives $\begin{bmatrix} 0 & 0 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} 0 & 0 \\ -8 & 4 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$, the general solution is $x_1 = \frac{1}{2}s$, $x_2 = s$, or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s \\ s \end{bmatrix} = s\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$, so $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 3$.

$\lambda = -1$ gives $\begin{bmatrix} -4 & 0 \\ -8 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since $\begin{bmatrix} -4 & 0 \\ -8 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the general solution is $x_1 = 0$, $x_2 = s$, or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ s \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = -1$.

(b) $\begin{bmatrix} \lambda-10 & 9 \\ -4 & \lambda+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 4$ gives $\begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}$, the general solution is $x_1 = \frac{3}{2}s$, $x_2 = s$, or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}s \\ s \end{bmatrix} = s\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$, so $\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 4$.

(c) $\begin{bmatrix} \lambda & -3 \\ -4 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = \sqrt{12}$ gives $\begin{bmatrix} \sqrt{12} & -3 \\ -4 & \sqrt{12} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} \sqrt{12} & -3 \\ -4 & \sqrt{12} \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & -\frac{3}{\sqrt{12}} \\ 0 & 0 \end{bmatrix}$, the general solution is $x_1 = \frac{3}{\sqrt{12}}s$, $x_2 = s$, or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{12}}s \\ s \end{bmatrix} = s\begin{bmatrix} \frac{3}{\sqrt{12}} \\ 1 \end{bmatrix}$, so $\begin{bmatrix} \frac{3}{\sqrt{12}} \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = \sqrt{12}$.

$\lambda = -\sqrt{12}$ gives $\begin{bmatrix} -\sqrt{12} & -3 \\ -4 & -\sqrt{12} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} -\sqrt{12} & -3 \\ -4 & -\sqrt{12} \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & \frac{3}{\sqrt{12}} \\ 0 & 0 \end{bmatrix}$, the general solution is $x_1 = -\frac{3}{\sqrt{12}}s$, $x_2 = s$, or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{\sqrt{12}}s \\ s \end{bmatrix} = s\begin{bmatrix} -\frac{3}{\sqrt{12}} \\ 1 \end{bmatrix}$, so $\begin{bmatrix} -\frac{3}{\sqrt{12}} \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = -\sqrt{12}$.

(d) There are no eigenspaces since there are no real eigenvalues.

(e) $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 0$ gives $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Clearly, every vector in R^2 is a solution, so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 0$.

(f) $\begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 1$ gives $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Clearly, every vector in R^2 is a solution, so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 1$.

7. (a) The eigenvalues satisfy $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ or $(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$, so the eigenvalues are $\lambda = 1, 2, 3$.

(b) The eigenvalues satisfy $\lambda^3 - 2\lambda = 0$ or $\lambda\left(\lambda + \sqrt{2}\right)\left(\lambda - \sqrt{2}\right) = 0$, so the eigenvalues are $\lambda = 0, -\sqrt{2}, \sqrt{2}$.

(c) The eigenvalues satisfy $\lambda^3 + 8\lambda^2 + \lambda + 8 = 0$ or $(\lambda + 8)(\lambda^2 + 1) = 0$, so the only (real) eigenvalue is $\lambda = -8$.

(d) The eigenvalues satisfy $\lambda^3 - \lambda^2 - \lambda - 2 = 0$ or $(\lambda - 2)(\lambda^2 + \lambda + 1) = 0$, so the only (real eigenvalue is $\lambda = 2$.

(e) The eigenvalues satisfy $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$ or $(\lambda - 2)^3 = 0$, so the only eigenvalue is $\lambda = 2$.

(f) The eigenvalues satisfy $\lambda^3 - 2\lambda^2 - 15\lambda + 36 = 0$ or $(\lambda + 4)(\lambda - 3)^2 = 0$, so the eigenvalues are $\lambda = -4, 3$.

9. (a) $\det\begin{bmatrix} \lambda & 0 & -2 & 0 \\ -1 & \lambda & -1 & 0 \\ 0 & -1 & \lambda+2 & 0 \\ 0 & 0 & 0 & \lambda-1 \end{bmatrix}$

$= (\lambda - 1)\det\begin{bmatrix} \lambda & 0 & -2 \\ -1 & \lambda & -1 \\ 0 & -1 & \lambda+2 \end{bmatrix}$

$= (\lambda - 1)\{\lambda[\lambda(\lambda + 2) - 1] - 2(1 - 0)\}$

$= (\lambda - 1)(\lambda^3 + 2\lambda^2 - \lambda - 2)$

$= \lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2$

The characteristic equation is $\lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2 = 0$.

(b) $\det\begin{bmatrix}\lambda-10 & 9 & 0 & 0\\ -4 & \lambda+2 & 0 & 0\\ 0 & 0 & \lambda+2 & 7\\ 0 & 0 & -1 & \lambda-2\end{bmatrix} = (\lambda-10)\det\begin{bmatrix}\lambda+2 & 0 & 0\\ 0 & \lambda+2 & 7\\ 0 & -1 & \lambda-2\end{bmatrix} - 9\det\begin{bmatrix}-4 & 0 & 0\\ 0 & \lambda+2 & 7\\ 0 & -1 & \lambda-2\end{bmatrix}$

$$\begin{aligned}&= (\lambda-10)(\lambda+2)[(\lambda+2)(\lambda-2)+7] - 9(-4)[(\lambda+2)(\lambda-2)+7]\\ &= (\lambda^2-8\lambda-20+36)(\lambda^2-4+7)\\ &= (\lambda^2-8\lambda+16)(\lambda^2+3)\\ &= \lambda^4-8\lambda^3+19\lambda^2-24\lambda+48\end{aligned}$$

The characteristic equation is $\lambda^4-8\lambda^3+19\lambda^2-24\lambda+48=0$.

11. (a) $\begin{bmatrix}\lambda & 0 & -2 & 0\\ -1 & \lambda & -1 & 0\\ 0 & -1 & \lambda+2 & 0\\ 0 & 0 & 0 & \lambda-1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

$\lambda = 1$ gives $\begin{bmatrix}1 & 0 & -2 & 0\\ -1 & 1 & -1 & 0\\ 0 & -1 & 3 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

Since $\begin{bmatrix}1 & 0 & -2 & 0\\ -1 & 1 & -1 & 0\\ 0 & -1 & 3 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ reduces to $\begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & -3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$, the general solution is $x_1 = 2s$, $x_2 = 3s$, $x_3 = s$,

$x_4 = t$ or $\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}2s\\ 3s\\ s\\ t\end{bmatrix} = s\begin{bmatrix}2\\3\\1\\0\end{bmatrix} + t\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, so $\begin{bmatrix}2\\3\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 1$.

$\lambda = -2$ gives $\begin{bmatrix}-2 & 0 & -2 & 0\\ -1 & -2 & -1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

Since $\begin{bmatrix}-2 & 0 & -2 & 0\\ -1 & -2 & -1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & -3\end{bmatrix}$ reduces to $\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$, the general solution is $x_1 = -s$, $x_2 = 0$, $x_3 = s$,

$x_4 = 0$ or $\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}-s\\ 0\\ s\\ 0\end{bmatrix} = s\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$, so $\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = -2$.

$\lambda = -1$ gives $\begin{bmatrix}-1 & 0 & -2 & 0\\ -1 & -1 & -1 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

Since $\begin{bmatrix} -1 & 0 & -2 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ reduces to

$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the general solution is

$x_1 = -2s,\ \ x_2 = s,\ \ x_3 = s,\ \ x_4 = 0$ or

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ s \\ 0 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ is a basis

for the eigenspace corresponding to $\lambda = -1$.

(b) $\begin{bmatrix} \lambda-10 & 9 & 0 & 0 \\ -4 & \lambda+2 & 0 & 0 \\ 0 & 0 & \lambda+2 & 7 \\ 0 & 0 & -1 & \lambda-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 4$ gives $\begin{bmatrix} -6 & 9 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ 0 & 0 & 6 & -7 \\ 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} -6 & 9 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ 0 & 0 & 6 & -7 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ reduces to

$\begin{bmatrix} 1 & -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the general solution is

$x_1 = \frac{3}{2}s,\ \ x_2 = s,\ \ x_3 = 0,\ \ x_4 = 0$ or

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}s \\ s \\ 0 \\ 0 \end{bmatrix} = s\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis for

the eigenspace corresponding to $\lambda = 4$.

13. Since A is upper triangular, the eigenvalues of A are the diagonal entries: $1, \frac{1}{2}, 0, 2$. Thus the eigenvalues of A^9 are $1^9 = 1$, $\left(\frac{1}{2}\right)^9 = \frac{1}{512}$, $0^9 = 0$, and $2^9 = 512$.

15. If a line is invariant under A, then it can be expressed in terms of an eigenvector of A.

(a)
$$\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & 1 \\ -2 & \lambda-1 \end{vmatrix} = (\lambda-4)(\lambda-1)+2 = \lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2)$$

$\begin{bmatrix} \lambda-4 & 1 \\ -2 & \lambda-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 3$ gives $\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, so a general solution is $x = y$. The line $y = x$ is invariant under A. $\lambda = 2$ gives $\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$ can be reduced to $\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$, so a general solution is $2x = y$. The line $y = 2x$ is invariant under A.

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1$

Since the characteristic equation $\lambda^2 + 1 = 0$ has no real solutions, there are no lines that are invariant under A.

(c) $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & -3 \\ 0 & \lambda-2 \end{vmatrix} = (\lambda-2)^2$

$\begin{bmatrix} \lambda-2 & -3 \\ 0 & \lambda-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 2$ gives $\begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, the general solution is $y = 0$, and the line $y = 0$ is invariant under A.

23. We have that $A\mathbf{x} = \lambda\mathbf{x}$.

$A^{-1}(A\mathbf{x}) = I\mathbf{x} = \mathbf{x}$, but also

$A^{-1}(A\mathbf{x}) = A^{-1}(\lambda\mathbf{x}) = \lambda(A^{-1}\mathbf{x})$ so $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$.

Thus $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}, with corresponding eigenvector $\mathbf{x}$.

25. We have that $A\mathbf{x} = \lambda\mathbf{x}$.
Since s is a scalar, $(sA)\mathbf{x} = s(A\mathbf{x}) = s(\lambda\mathbf{x}) = (s\lambda)\mathbf{x}$ so $s\lambda$ is an eigenvalue of sA with corresponding eigenvector $\mathbf{x}$.

True/False 5.1

(a) False; the vector $\mathbf{x}$ must be nonzero to be an eigenvector—if $\mathbf{x} = \mathbf{0}$, then $A\mathbf{x} = \lambda\mathbf{x}$ for all λ.

(b) False; if λ is an eigenvalue of A, then $\det(\lambda I - A) = 0$, so $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

(c) True; since $p(0) = 0^2 + 1 \neq 0$, $\lambda = 0$ is not an eigenvalue of A and A is invertible by Theorem 5.1.5.

(d) False; the eigenspace corresponding to λ contains the vector $\mathbf{x} = \mathbf{0}$, which is not an eigenvector of A.

(e) True; if 0 is an eigenvalue of A, then $0^2 = 0$ is an eigenvalue of A^2, so A^2 is singular.

(f) False; the reduced row echelon form of an invertible $n \times n$ matrix A is I_n which has only one eigenvalue, $\lambda = 1$, whereas A can have eigenvalues that are not 1. For instance, the matrix $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ has the eigenvalues $\lambda = 3$ and $\lambda = -1$.

(g) False; if 0 is an eigenvalue of A, then A is singular so the set of column vectors of A cannot be linearly independent.

Section 5.2

Exercise Set 5.2

1. Since the determinant is a similarity invariant and $\det(A) = 2 - 3 = -1$ while $\det(B) = -2 - 0 = -2$, A and B are not similar matrices.

3. A reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ while B reduces to $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so they have different ranks. Since rank is a similarity invariant, A and B are not similar matrices.

5. Since the geometric multiplicity of an eigenvalue is less than or equal to its algebraic multiplicity, the eigenspace for $\lambda = 0$ can have dimension 1 or 2, the eigenspace for $\lambda = 1$ has dimension 1, and the eigenspace for $\lambda = 2$ can have dimension 1, 2, or 3.

7. Since the matrix is lower triangular, with 2s on the diagonal, the only eigenvalue is $\lambda = 2$.
For $\lambda = 2$, $2I - A = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, which has rank 1, hence nullity 1. Thus, the matrix has only 1 distinct eigenvector and is not diagonalizable.

9. Since the matrix is lower triangular with diagonal entries of 3, 2, and 2, the eigenvalues are 3 and 2, where 2 has algebraic multiplicity 2.
For $\lambda = 3$, $3I - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ which has rank 2, hence nullity 1, so the eigenspace corresponding to $\lambda = 3$ has dimension 1.
For $\lambda = 2$, $2I - A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ which has rank 2, hence nullity 1, so the eigenspace corresponding to $\lambda = 2$ has dimension 1.
The matrix is not diagonalizable, since it has only 2 distinct eigenvectors.

11. Since the matrix is upper triangular with diagonal entries 2, 2, 3, and 3, the eigenvalues are 2 and 3 each with algebraic multiplicity 2.
For $\lambda = 2$, $2I - A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ which has rank 3, hence nullity 1, so the eigenspace corresponding to $\lambda = 2$ has dimension 1.
For $\lambda = 3$, $3I - A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which has rank 3, hence nullity 1, so the eigenspace corresponding to $\lambda = 3$ has dimension 1.
The matrix is not diagonalizable since it has only 2 distinct eigenvectors. Note that showing that the geometric multiplicity of either eigenvalue is less than its algebraic multiplicity is sufficient to show that the matrix is not diagonalizable.

13. A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ (the diagonal entries, since A is triangular).

$(\lambda I - A)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} \lambda-1 & 0 \\ -6 & \lambda+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\lambda_1 = 1$ gives $\begin{bmatrix} 0 & 0 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} 0 & 0 \\ -6 & 2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$, the general solution is $x_1 = \frac{1}{3}s$, $x_2 = s$, or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$, and $\mathbf{p}_1 = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = 1$.

$\lambda_2 = -1$ gives $\begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the general solution is $x_1 = 0$, $x_2 = s$ or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_2 = -1$.

Thus, $P = [\mathbf{p}_1 \quad \mathbf{p}_2] = \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{bmatrix}$ diagonalizes A.

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 3 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

15. A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ (the diagonal entries, since A is triangular).

$(\lambda I - A)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} \lambda-2 & 0 & 2 \\ 0 & \lambda-3 & 0 \\ 0 & 0 & \lambda-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\lambda_1 = 2$ gives $\begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ reduces to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = s$, $x_2 = 0$, $x_3 = 0$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = 2$.

$\lambda_2 = 3$ gives $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ which has general solution $x_1 = -2s$, $x_2 = t$, $x_3 = s$, or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{p}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_2 = 3$. Thus $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ diagonalizes A.

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{aligned}$$

17. $$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda+1 & -4 & 2 \\ 3 & \lambda-4 & 0 \\ 3 & -1 & \lambda-3 \end{vmatrix} \\ &= x^3 - 6x^2 + 11x - 6 \\ &= (x-1)(x-2)(x-3) \end{aligned}$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$, each with algebraic multiplicity 1. Since each eigenvalue must have nonzero geometric multiplicity, the geometric multiplicity of each eigenvalue is also 1 and A is diagonalizable.

$(\lambda I - A)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} \lambda+1 & -4 & 2 \\ 3 & \lambda-4 & 0 \\ 3 & -1 & \lambda-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\lambda_1 = 1$ gives $\begin{bmatrix} 2 & -4 & 2 \\ 3 & -3 & 0 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Since $\begin{bmatrix} 2 & -4 & 2 \\ 3 & -3 & 0 \\ 3 & -1 & -2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = s,\ x_2 = s,\ x_3 = s$, or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = 1$.

$\lambda_2 = 2$ gives $\begin{bmatrix} 3 & -4 & 2 \\ 3 & -2 & 0 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Since $\begin{bmatrix} 3 & -4 & 2 \\ 3 & -2 & 0 \\ 3 & -1 & -1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = \frac{2}{3}s,\ x_2 = s,\ x_3 = s$ or $x_1 = 2t,\ x_2 = 3t,\ x_3 = 3t$ which is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_2 = 2$.

$\lambda_3 = 3$ gives $\begin{bmatrix} 4 & -4 & 2 \\ 3 & -1 & 0 \\ 3 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Since $\begin{bmatrix} 4 & -4 & 2 \\ 3 & -1 & 0 \\ 3 & -1 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix}$, a general solution is $x_1 = \frac{1}{4}s,\ x_2 = \frac{3}{4}s,\ x_3 = s$ or $x_1 = t,\ x_2 = 3t,\ x_3 = 4t$ which is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_3 = 3$.

Thus $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ diagonalizes A and

$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$

19. Since A is triangular with diagonal entries of 0, 0, 1, the eigenvalues are $\lambda_1 = 0$ (with algebraic multiplicity 2) and $\lambda_2 = 1$.

$\lambda_2 = 1$ must also have geometric multiplicity 1.

$(\lambda I - A)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ -3 & 0 & \lambda - 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

$\lambda_1 = 0$ gives $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Since $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & -1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, a general solution is $x_1 = -\frac{1}{3}r,\ x_2 = t,\ x_3 = r$ or $x_1 = s,\ x_2 = t,\ x_3 = -3s$ which is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Thus $\lambda_1 = 0$ has geometric multiplicity 2, and A is diagonalizable.

$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix},\ \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are a basis for the eigenspace corresponding to $\lambda_1 = 0$.

$\lambda_2 = 1$ gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Since $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, a general solution is $x_1 = 0,\ x_2 = 0,\ x_3 = s$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_2 = 1$.

Thus $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

diagonalizes A and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

21. Since A is triangular with entries −2, −2, 3, and 3 on the diagonal, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 3$, both with algebraic multiplicity 2.

$(\lambda I - A)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} \lambda+2 & 0 & 0 & 0 \\ 0 & \lambda+2 & -5 & 5 \\ 0 & 0 & \lambda-3 & 0 \\ 0 & 0 & 0 & \lambda-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\lambda_1 = -2$ gives $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

Since $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$ reduces to

$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, a general solution is $x_1 = s$,

$x_2 = t$, $x_3 = 0$, $x_4 = 0$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$

Thus $\lambda_1 = -2$ has geometric multiplicity 2, and

$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ are a basis for the eigenspace corresponding to $\lambda_1 = -2$.

$\lambda_2 = 3$ gives $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

Since $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ reduces to

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, a general solution is $x_1 = 0$,

$x_2 = s - t$, $x_3 = s$, $x_4 = t$, or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $\lambda_2 = 3$ has geometric multiplicity 2, and

$\mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{p}_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ are a basis for the eigenspace corresponding to $\lambda_2 = 3$.

Since the geometric and algebraic multiplicities are equal for both eigenvalues, A is diagonalizable.

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3 \quad \mathbf{p}_4] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

diagonalizes A, and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

23. $$\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & -7 & 1 \\ 0 & \lambda-1 & 0 \\ 0 & -15 & \lambda+2 \end{vmatrix} = \lambda^3 + 2\lambda^2 - \lambda - 2 = (\lambda+2)(\lambda+1)(\lambda-1)$$

Thus, the eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = -1$, and $\lambda_3 = 1$.

$(\lambda I - A)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} \lambda+1 & -7 & 1 \\ 0 & \lambda-1 & 0 \\ 0 & -15 & \lambda+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\lambda_1 = -2$ gives $\begin{bmatrix} -1 & -7 & 1 \\ 0 & -3 & 0 \\ 0 & -15 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Since $\begin{bmatrix} -1 & -7 & 1 \\ 0 & -3 & 0 \\ 0 & -15 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, a general solution is $x_1 = s,\ x_2 = 0,\ x_3 = s,$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, so $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = -2.$

$\lambda_2 = -1$ gives $\begin{bmatrix} 0 & -7 & 1 \\ 0 & -2 & 0 \\ 0 & -15 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Since $\begin{bmatrix} 0 & -7 & 1 \\ 0 & -2 & 0 \\ 0 & -15 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, a general solution is $x_1 = s,\ x_2 = 0,\ x_3 = 0,$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ so $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_2 = -1.$

$\lambda_3 = 1$ gives $\begin{bmatrix} 2 & -7 & 1 \\ 0 & 0 & 0 \\ 0 & -15 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Since $\begin{bmatrix} 2 & -7 & 1 \\ 0 & 0 & 0 \\ 0 & -15 & 3 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}$, a general solution is $x_1 = \frac{1}{5}s,\ x_2 = \frac{1}{5}s,\ x_3 = s$ or $x_1 = t,\ x_2 = t,\ x_3 = 5t$ which is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ so $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_3 = 1.$

$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}$ diagonalizes A, and $D = P^{-1}AP$

$$\begin{aligned} &= \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A^{11} = PD^{11}P^{-1} &= P\begin{bmatrix} (-2)^{11} & 0 & 0 \\ 0 & (-1)^{11} & 0 \\ 0 & 0 & 1^{11} \end{bmatrix}P^{-1} \\ &= P\begin{bmatrix} -2048 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}P^{-1} \\ &= \begin{bmatrix} -1 & 10,237 & -2047 \\ 0 & 1 & 0 \\ 0 & 10,245 & -2048 \end{bmatrix} \end{aligned}$$

25. $\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-3 \end{vmatrix}$
$= \lambda^3 - 8\lambda^2 + 19\lambda - 12$
$= (\lambda-1)(\lambda-3)(\lambda-4)$

The eigenvalues of A are $\lambda_1 = 1,\ \lambda_2 = 3,$ and $\lambda_3 = 4.$

$(\lambda I - A)\mathbf{x} = \mathbf{0}$ is

$\begin{bmatrix} \lambda-3 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

$\lambda_1 = 1$ gives $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Since $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, a general solution is $x_1 = s,\ x_2 = 2s,\ x_3 = s$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = 1.$

$\lambda_2 = 3$ gives $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Since $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = s$, $x_2 = 0$, $x_3 = -s$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_2 = 3$.

$\lambda_3 = 4$ gives $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, a general solution is $x_1 = s$, $x_2 = -s$, $x_3 = s$, or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{p}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_3 = 4$.

Thus $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ diagonalizes A and

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

$$P^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$A^n = PD^nP^{-1}$$
$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix}\begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

27. Using the result in Exercise 20 of Section 5.1, one possibility is $P = \begin{bmatrix} -b & -b \\ a-\lambda_1 & a-\lambda_2 \end{bmatrix}$ where

$\lambda_1 = \frac{1}{2}\left[a + d + \sqrt{(a-d)^2 + 4bc}\right]$ and

$\lambda_2 = \frac{1}{2}\left[a + d - \sqrt{(a-d)^2 + 4bc}\right]$.

31. If A is diagonalizable, then $P^{-1}AP = D$. For the same P, $P^{-1}A^kP$

$$= \underbrace{P^{-1}A(PP^{-1})A(PP^{-1})A\cdots(PP^{-1})AP}_{\text{with } k \text{ factors of } A}$$
$$= \underbrace{(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)}_{k \text{ factors of } P^{-1}AP}$$
$$= D^k$$

Since D is a diagonal matrix, so is D^k, so A^k is also diagonalizable.

33. **(a)** The dimension of the eigenspace corresponding to an eigenvalue is the geometric multiplicity of the eigenvalue, which must be less than or equal to the algebraic multiplicity of the eigenvalue. Thus, for $\lambda = 1$, the dimension is 1; for $\lambda = 3$, the dimension is 1 or 2; for $\lambda = 4$, the dimension is 1, 2, or 3.

(b) If A is diagonalizable then each eigenvalue has the same algebraic and geometric multiplicity. Thus the dimensions of the eigenspaces are:
$\lambda = 1$: dimension 1
$\lambda = 3$: dimension 2
$\lambda = 4$: dimension 3

(c) Since only $\lambda = 4$ can have an eigenspace with dimension 3, the eigenvalue must be 4.

True/False 5.2

(a) True; use $P = I$.

(b) True; since A is similar to B, $B = P_1^{-1}AP_1$ for some invertible matrix P_1. Since B is similar to C, $C = P_2^{-1}BP_2$ for some invertible matrix P_2.

Then $C = P_2^{-1}BP_2$
$$= P_2^{-1}(P_1^{-1}AP_1)P_2$$
$$= (P_2^{-1}P_1^{-1})A(P_1P_2)$$
$$= (P_1P_2)^{-1}A(P_1P_2).$$

Since the product of invertible matrices is invertible, $P = P_1P_2$ is invertible and A is similar to C.

(c) True; since $B = P^{-1}AP$, then $B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}P$ so A^{-1} and B^{-1} are similar.

(d) False; a matrix P that diagonalizes A is not unique—it depends on the order of the eigenvalues in the diagonal matrix D and the choice of basis for each eigenspace.

(e) True; if $P^{-1}AP = D$, then $P^{-1}A^{-1}P = (P^{-1}AP)^{-1} = D^{-1}$, so A^{-1} is also diagonalizable.

(f) True; $D^T = (P^{-1}AP)^T = P^T A^T (P^{-1})^T$, so A^T is also diagonalizable.

(g) True; if A is $n \times n$ and has n linearly independent eigenvectors, then the geometric multiplicity of each eigenvalue must be equal to its algebraic multiplicity, hence A is diagonalizable.

(h) True; if every eigenvalue of A has algebraic multiplicity 1, then every eigenvalue also has geometric multiplicity 1. Since the algebraic and geometric multiplicities are the same for each eigenvalue, A is diagonalizable.

Section 5.3

Exercise Set 5.3

1. $\bar{\mathbf{u}} = \left(\overline{2-i}, \overline{4i}, \overline{1+i}\right) = (2+i, -4i, 1-i)$

Re($\mathbf{u}$) = (Re(2 − i), Re(4i), Re(1 + i)) = (2, 0, 1)

Im($\mathbf{u}$) = (Im(2 − i), Im(4i), Im(1 + i)) = (−1, 4, 1)

$$\begin{aligned}\|\mathbf{u}\| &= \sqrt{|2-i|^2 + |4i|^2 + |1+i|^2} \\ &= \sqrt{\left(\sqrt{2^2+1^2}\right)^2 + \left(\sqrt{4^2}\right)^2 + \left(\sqrt{1^2+1^2}\right)^2} \\ &= \sqrt{5+16+2} \\ &= \sqrt{23}\end{aligned}$$

5. If $i\mathbf{x} - 3\mathbf{v} = \bar{\mathbf{u}}$, then $i\mathbf{x} = 3\mathbf{v} + \bar{\mathbf{u}}$, and

$$\mathbf{x} = \frac{1}{i}(3\mathbf{v} + \bar{\mathbf{u}}) = -i(3\mathbf{v} + \bar{\mathbf{u}}).$$

$$\begin{aligned}\mathbf{x} &= -i(3(1+i, 2-i, 4) + \overline{(3-4i, 2+i, -6i)}) \\ &= -i((3+3i, 6-3i, 12) + (3+4i, 2-i, 6i)) \\ &= -i(6+7i, 8-4i, 12+6i) \\ &= (7-6i, -4-8i, 6-12i)\end{aligned}$$

7. $\bar{A} = \begin{bmatrix} \overline{-5i} & \bar{4} \\ \overline{2-i} & \overline{1+5i} \end{bmatrix} = \begin{bmatrix} 5i & 4 \\ 2+i & 1-5i \end{bmatrix}$

$$\text{Re}(A) = \begin{bmatrix} \text{Re}(-5i) & \text{Re}(4) \\ \text{Re}(2-i) & \text{Re}(1+5i) \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 2 & 1 \end{bmatrix}$$

$$\text{Im}(A) = \begin{bmatrix} \text{Im}(-5i) & \text{Im}(4) \\ \text{Im}(2-i) & \text{Im}(1+5i) \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ -1 & 5 \end{bmatrix}$$

$$\begin{aligned}\det(A) &= -5i(1+5i) - 4(2-i) \\ &= -5i + 25 - 8 + 4i \\ &= 17 - i\end{aligned}$$

tr(A) = −5i + (1 + 5i) = 1

11.
$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (i, 2i, 3)\cdot(4, -2i, 1+i) \\ &= i(\bar{4}) + 2i(\overline{-2i}) + 3(\overline{1+i}) \\ &= 4i + 2i(2i) + 3(1-i) \\ &= 4i - 4 + 3 - 3i \\ &= -1 + i\end{aligned}$$

$$\begin{aligned}\mathbf{u}\cdot\mathbf{w} &= (i, 2i, 3)\cdot(2-i, 2i, 5+3i) \\ &= i(\overline{2-i}) + 2i(\overline{2i}) + 3(\overline{5+3i}) \\ &= i(2+i) + 2i(-2i) + 3(5-3i) \\ &= 2i - 1 + 4 + 15 - 9i \\ &= 18 - 7i\end{aligned}$$

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= (4, -2i, 1+i)\cdot(2-i, 2i, 5+3i) \\ &= 4(\overline{2-i}) - 2i(\overline{2i}) + (1+i)(\overline{5+3i}) \\ &= 4(2+i) - 2i(-2i) + (1+i)(5-3i) \\ &= 8 + 4i - 4 + 5 + 2i + 3 \\ &= 12 + 6i\end{aligned}$$

$$\begin{aligned}\mathbf{v}\cdot\mathbf{u} &= 4(-i) - 2i(-2i) + (1+i)(3) \\ &= -4i - 4 + 3 + 3i \\ &= -1 - i \\ &= \overline{\mathbf{u}\cdot\mathbf{v}}\end{aligned}$$

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) &= (i, 2i, 3)\cdot(6-i, 0, 6+4i) \\ &= i(6+i) + 2i(0) + 3(6-4i) \\ &= 6i - 1 + 18 - 12i \\ &= 17 - 6i \\ &= (-1+i) + (18-7i) \\ &= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}\end{aligned}$$

k($\mathbf{u}$ · $\mathbf{v}$) = 2i(−1 + i) = −2i − 2 = −2 − 2i

$$\begin{aligned}(k\mathbf{u})\cdot\mathbf{v} &= (2i(i, 2i, 3))\cdot(4, -2i, 1+i) \\ &= (-2, -4, 6i)\cdot(4, -2i, 1+i) \\ &= -2(4) - 4(2i) + 6i(1-i) \\ &= -8 - 8i + 6i + 6 \\ &= -2 - 2i \\ &= k(\mathbf{u}\cdot\mathbf{v})\end{aligned}$$

13.
$$\begin{aligned}\mathbf{u}\cdot\bar{\mathbf{v}} &= (i, 2i, 3)\cdot(4, 2i, 1-i) \\ &= i(4) + 2i(-2i) + 3(1+i) \\ &= 4i + 4 + 3 + 3i \\ &= 7 + 7i\end{aligned}$$

$$\overline{\mathbf{w}\cdot\mathbf{u}} = \overline{(\overline{\mathbf{u}\cdot\mathbf{w}})} = \mathbf{u}\cdot\mathbf{w} = 18 - 7i$$

$$\begin{aligned}\overline{(\mathbf{u}\cdot\bar{\mathbf{v}}) - \overline{\mathbf{w}\cdot\mathbf{u}}} &= \overline{7 + 7i - (18 - 7i)} \\ &= \overline{-11 + 14i} \\ &= -11 - 14i\end{aligned}$$

15. $\operatorname{tr}(A) = 4 + 0 = 4$
$\det(A) = 4(0) - 1(-5) = 5$
The characteristic equation of A is
$\lambda^2 - 4\lambda + 5 = 0$, which has solutions $\lambda = 2 \pm i$.
$(\lambda I - A) = \mathbf{0}$ is $\begin{bmatrix} \lambda-4 & 5 \\ -1 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
$\lambda_1 = 2 - i$ gives $\begin{bmatrix} -2-i & 5 \\ -1 & 2-i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
Since $\begin{bmatrix} -2-i & 5 \\ -1 & 2-i \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & -2+i \\ 0 & 0 \end{bmatrix}$, a general solution is $x_1 = (2-i)s$, $x_2 = s$ or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$ so $\mathbf{x}_1 = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = 2 - i$.
$\mathbf{x}_2 = \overline{\mathbf{x}}_1 = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_2 = \overline{\lambda}_1 = 2 + i$.

17. $\operatorname{tr}(A) = 5 + 3 = 8$
$\det(A) = 5(3) - 1(-2) = 17$
The characteristic equation of A is
$\lambda^2 - 8\lambda + 17 = 0$ which has solutions $\lambda = 4 \pm i$.
$(\lambda I - A)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} \lambda-5 & 2 \\ -1 & \lambda-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
$\lambda_1 = 4 - i$ gives $\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
Since $\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & -1+i \\ 0 & 0 \end{bmatrix}$, a general solution is $x_1 = (1-i)s$, $x_2 = s$ or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ so $\mathbf{x}_1 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = 4 - i$.
$\mathbf{x}_2 = \overline{\mathbf{x}}_1 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_2 = \overline{\lambda}_1 = 4 + i$.

19. Here $a = b = 1$. The angle from the positive x-axis to the ray that joins the origin to the point $(1, 1)$ is $\phi = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$.
$|\lambda| = |1 + 1i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

21. Here $a = 1$, $b = -\sqrt{3}$. The angle from the positive x-axis to the ray that joins the origin to the point $(1, -\sqrt{3})$ is $\phi = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$.
$|\lambda| = |1 - i\sqrt{3}| = \sqrt{1+3} = 2$

23. $\operatorname{tr}(A) = -1 + 7 = 6$
$\det(A) = -1(7) - 4(-5) = 13$
The characteristic equation of A is
$\lambda^2 - 6\lambda + 13 = 0$ which has solutions $\lambda = 3 \pm 2i$.
For $\lambda = 3 - 2i$, $\det(\lambda I - A)\mathbf{x} = \mathbf{0}$ is
$\begin{bmatrix} 4-2i & 5 \\ -4 & -4-2i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since
$\begin{bmatrix} 4-2i & 5 \\ -4 & -4-2i \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 1+\frac{1}{2}i \\ 0 & 0 \end{bmatrix}$, a general solution is $x_1 = \left(-1 - \frac{1}{2}i\right)s$, $x_2 = s$ or
$x_1 = (-2-i)t$, $x_2 = 2t$ which is
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -2-i \\ 2 \end{bmatrix}$ so $\mathbf{x} = \begin{bmatrix} -2-i \\ 2 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 3 - 2i$.
$\operatorname{Re}(\mathbf{x}) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $\operatorname{Im}(\mathbf{x}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.
Thus $A = PCP^{-1}$ where $P = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix}$ and
$C = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$.

25. $\operatorname{tr}(A) = 8 + 2 = 10$
$\det(A) = 8(2) - (-3)(6) = 34$
The characteristic equation of A is
$\lambda^2 - 10\lambda + 34 = 0$ which has solutions
$\lambda = 5 \pm 3i$.
For $\lambda = 5 - 3i$, $(\lambda I - A)\mathbf{x} = \mathbf{0}$ is
$\begin{bmatrix} -3-3i & -6 \\ 3 & 3-3i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
Since $\begin{bmatrix} -3-3i & -6 \\ 3 & 3-3i \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix}$, a general solution is $x_1 = (-1+i)s$, $x_2 = s$ or
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} 1-i \\ -1 \end{bmatrix}$, so $\mathbf{x} = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 5 - 3i$.
$\operatorname{Re}(\mathbf{x}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\operatorname{Im}(\mathbf{x}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Thus, $A = PCP^{-1}$ where $P = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & -3 \\ 3 & 5 \end{bmatrix}$.

27. **(a)** $\mathbf{u} \cdot \mathbf{v} = (2i, i, 3i) \cdot (i, 6i, k)$
$= 2i(\overline{-i}) + i(\overline{-6i}) + 3i(\overline{k})$
$= 2 + 6 + 3i\overline{k}$
$= 8 + 3i\overline{k}$

If $\mathbf{u} \cdot \mathbf{v} = 0$, then $3i\overline{k} = -8$ or $\overline{k} = -\frac{8}{3i} = \frac{8}{3}i$ so $k = \frac{8}{3}i$.

(b) $\mathbf{u} \cdot \mathbf{v} = (k, k, 1+i) \cdot (1, -1, 1-i)$
$= k(1) + k(-1) + (1+i)(1+i)$
$= (1+i)^2$
$= 2i$

Thus, there is no complex scalar k for which $\mathbf{u} \cdot \mathbf{v} = 0$.

True/False 5.3

(a) False; non-real complex eigenvalues occur in conjugate pairs, so there must be an even number. Since a real 5×5 matrix will have 5 eigenvalues, at least one must be real.

(b) True; $\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0$ is the characteristic equation of a 2×2 matrix A.

(c) False; if A is a matrix with real entries and complex eigenvalues and k is a real scalar, $k \neq 0, 1$, then kA has the same eigenvalues as A with the same algebraic multiplicities but $\text{tr}(kA) = k\text{tr}(A) \neq \text{tr}(A)$.

(d) True; complex eigenvalues and eigenvectors occur in conjugate pairs.

(e) False; real symmetric matrices have real eigenvalues.

(f) False; this is only true if the eigenvalues of A satisfy $|\lambda| = 1$.

Section 5.4

Exercise Set 5.4

1. **(a)** The coefficient matrix for the system is $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

$\det(\lambda I - A) = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$

The eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -1$.

$(\lambda I - A)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\lambda_1 = 5$ gives $\begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ a general solution is $x_1 = s$, $x_2 = s$, or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = 5$.

$\lambda_2 = -1$ gives $\begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ a general solution is $x_1 = -2s$, $x_2 = s$ or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_2 = -1$.

Thus $P = [\mathbf{p}_1 \quad \mathbf{p}_2] = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$.

The substitutions $\mathbf{y} = P\mathbf{u}$ and $\mathbf{y}' = P\mathbf{u}'$ give the "diagonal system" $\mathbf{u}' = D\mathbf{u} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{u}$ or $\begin{matrix} u_1' = 5u_1 \\ u_2' = -u_2 \end{matrix}$ which has the solution $\mathbf{u} = \begin{bmatrix} c_1 e^{5x} \\ c_2 e^{-x} \end{bmatrix}$. Then $\mathbf{y} = P\mathbf{u}$ gives the solution

$\mathbf{y} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{5x} \\ c_2 e^{-x} \end{bmatrix} = \begin{bmatrix} c_1 e^{5x} - 2c_2 e^{-x} \\ c_1 e^{5x} + c_2 e^{-x} \end{bmatrix}$ and

the solution of the system is
$$\begin{aligned} y_1 &= c_1e^{5x} - 2c_2e^{-x} \\ y_2 &= c_1e^{5x} + c_2e^{-x} \end{aligned}.$$

(b) The initial conditions give $\begin{aligned} c_1 - 2c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}$ which has the solution $c_1 = c_2 = 0$, so the solution satisfying the given initial conditions is $\begin{aligned} y_1 &= 0 \\ y_2 &= 0 \end{aligned}$.

3. (a) The coefficient matrix for the system is
$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$
$$\begin{aligned} \det(\lambda I - A) &= \lambda^3 - 6\lambda^2 + 11\lambda - 6 \\ &= (\lambda-1)(\lambda-2)(\lambda-3) \end{aligned}$$
The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.

$(\lambda I - A)\mathbf{x} = \mathbf{0}$ is
$$\begin{bmatrix} \lambda-4 & 0 & -1 \\ 2 & \lambda-1 & 0 \\ 2 & 0 & \lambda-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
$\lambda_1 = 1$ gives $\begin{bmatrix} -3 & 0 & -1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} -3 & 0 & -1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = 0$, $x_2 = s$, $x_3 = 0$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = 1$.

$\lambda_2 = 2$ gives $\begin{bmatrix} -2 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} -2 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = -\frac{1}{2}s$, $x_2 = s$, $x_3 = s$ or $x_1 = -t$, $x_2 = 2t$, $x_3 = 2t$ which is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_2 = 2$.

$\lambda_3 = 3$ gives $\begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = s$, $x_2 = -s$, $x_3 = -s$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ and $\mathbf{p}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_3 = 3$.

Thus $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$ diagonalizes A and
$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$
The substitutions $\mathbf{y} = P\mathbf{u}$ and $\mathbf{y}' = P\mathbf{u}'$ give the "diagonal system"
$$\mathbf{u}' = D\mathbf{u} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{u} \text{ or } \begin{aligned} u_1' &= u_1 \\ u_2' &= 2u_2 \\ u_3' &= 3u_3 \end{aligned} \text{ which}$$
has the solution $\mathbf{u} = \begin{bmatrix} c_1e^{x} \\ c_2e^{2x} \\ c_3e^{3x} \end{bmatrix}$. Then $\mathbf{y} = P\mathbf{u}$ gives the solution

$$\mathbf{y} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} c_1e^x \\ c_2e^{2x} \\ c_3e^{3x} \end{bmatrix} = \begin{bmatrix} -c_2e^{2x} + c_3e^{3x} \\ c_1e^x + 2c_2e^{2x} - c_3e^{3x} \\ 2c_2e^{2x} - c_3e^{3x} \end{bmatrix}$$

and the solution of the system is

$$\begin{aligned} y_1 &= -c_2e^{2x} + c_3e^{3x} \\ y_2 &= c_1e^x + 2c_2e^{2x} - c_3e^{3x}. \\ y_3 &= 2c_2e^{2x} - c_3e^{3x} \end{aligned}$$

(b) The initial conditions give

$$\begin{aligned} -c_2 + c_3 &= -1 \\ c_1 + 2c_2 - c_3 &= 1 \\ 2c_2 - c_3 &= 0 \end{aligned}$$

which has the solution $c_1 = 1$, $c_2 = -1$, $c_3 = -2$, so the solution satisfying the given initial conditions is

$$\begin{aligned} y_1 &= e^{2x} - 2e^{3x} \\ y_2 &= e^x - 2e^{2x} + 2e^{3x}. \\ y_3 &= -2e^{2x} + 2e^{3x} \end{aligned}$$

5. Let $y = f(x)$ be a solution of $y' = ay$. Then for $g(x) = f(x)e^{-ax}$ we have

$$\begin{aligned} g'(x) &= f'(x)e^{-ax} + f(x)(-ae^{-ax}) \\ &= af(x)e^{-ax} - af(x)e^{-ax} \\ &= 0 \end{aligned}$$

Hence $g(x) = f(x)e^{-ax} = c$ where c is a constant or $f(x) = ce^{ax}$.

7. If $y_1 = y$ and $y_2 = y'$, then $y_2' = y'' = y' + 6y = y_2 + 6y_1$ which yields the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 6y_1 + y_2 \end{aligned}$$

The coefficient matrix of this system is $A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$.

$\det(\lambda I - A) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$

The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -2$.

$(\lambda I - A)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} \lambda & -1 \\ -6 & \lambda - 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\lambda_1 = 3$ gives $\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$, the general solution is $x_1 = \frac{1}{3}s$, $x_2 = s$ or $x_1 = t$, $x_2 = 3t$ which is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, so $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = 3$.

$\lambda_2 = -2$ gives $\begin{bmatrix} -2 & -1 \\ -6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since $\begin{bmatrix} -2 & -1 \\ -6 & -3 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$, the general solution is $x_1 = -\frac{1}{2}s$, $x_2 = s$ or $x_1 = t$, $x_2 = -2t$ which is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ so $\mathbf{p}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_2 = -2$. Thus $P = [\mathbf{p}_1 \quad \mathbf{p}_2] = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$ diagonalizes A and

$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$.

The substitutions $\mathbf{y} = P\mathbf{u}$ and $\mathbf{y}' = P\mathbf{u}'$ give the "diagonal system" $\mathbf{u}' = D\mathbf{u} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\mathbf{u}$ or

$$\begin{aligned} u_1' &= 3u_1 \\ u_2' &= -2u_2 \end{aligned}$$

which has the solution $\mathbf{u} = \begin{bmatrix} c_1e^{3x} \\ c_2e^{-2x} \end{bmatrix}$. Then $\mathbf{y} = P\mathbf{u}$ gives the solution

$$\mathbf{y} = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} c_1e^{3x} \\ c_2e^{-2x} \end{bmatrix} = \begin{bmatrix} c_1e^{3x} + c_2e^{-2x} \\ 3c_1e^{3x} - 2c_2e^{-2x} \end{bmatrix}.$$

Thus $y_1 = y = c_1e^{3x} + c_2e^{-2x}$ is the solution of the original differential equation.

9. Use the substitutions $y_1 = y$, $y_2 = y' = y_1'$ and $y_3 = y'' = y_2'$. This gives the system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}.$$

$$\det(\lambda I - A) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda-1)(\lambda-2)(\lambda-3)$$

The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.

$(\lambda I - A)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -6 & 11 & \lambda-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$\lambda_1 = 1$ gives $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -6 & 11 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -6 & 11 & -5 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = s$, $x_2 = s$, $x_3 = s$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ so $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = 1$.

$\lambda_2 = 2$ gives $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -6 & 11 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = \frac{1}{4}s$, $x_2 = \frac{1}{2}s$, $x_3 = s$ or $x_1 = t$, $x_2 = 2t$, $x_3 = 4t$ which is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ so $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_2 = 2$.

$\lambda_3 = 3$ gives $\begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & -1 \\ -6 & 11 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & -1 \\ -6 & 11 & -3 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -\frac{1}{9} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = \frac{1}{9}s$, $x_2 = \frac{1}{3}s$, $x_3 = s$ or $x_1 = t$, $x_2 = 3t$, $x_3 = 9t$ which is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ so $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_3 = 3$.

Thus $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ diagonalizes A and

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The substitutions $\mathbf{y} = P\mathbf{u}$ and $\mathbf{y}' = P\mathbf{u}'$ give the "diagonal system" $\mathbf{u}' = D\mathbf{u} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{u}$ or

$u_1' = u_1$
$u_2' = 2u_2$ which has the solution $\mathbf{u} = \begin{bmatrix} c_1e^x \\ c_2e^{2x} \\ c_3e^{3x} \end{bmatrix}$.
$u_3' = 3u_3$

Then $\mathbf{y} = P\mathbf{u}$ gives the solution

$$\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} c_1e^x \\ c_2e^{2x} \\ c_3e^{3x} \end{bmatrix} = \begin{bmatrix} c_1e^x + c_2e^{2x} + c_3e^{3x} \\ c_1e^x + 2c_2e^{2x} + 3c_3e^{3x} \\ c_1e^x + 4c_2e^{2x} + 9c_3e^{3x} \end{bmatrix}.$$

Thus $y_1 = y = c_1e^x + c_2e^{2x} + c_3e^{3x}$ is the solution of the original differential equation.

True/False 5.4

(a) False; just as not every system $A\mathbf{x} = \mathbf{b}$ has a solution, not every system of differential equations has a solution.

(b) False; $\mathbf{x}$ and $\mathbf{y}$ may differ by any constant vector $\mathbf{b}$.

(c) True

(d) True; if A is a square matrix with distinct real eigenvalues, it is diagonalizable.

(e) False; consider the case where A is diagonalizable but not diagonal. Then A is similar to a diagonal matrix P which can be used to solve the system $\mathbf{y}' = A\mathbf{y}$, but the systems $\mathbf{y}' = A\mathbf{y}$ and $\mathbf{u}' = P\mathbf{u}$ do not have the same solutions.

Chapter 5 Supplementary Exercises

1. (a) $\det(\lambda I - A) = \begin{vmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{vmatrix}$
$= \lambda^2 - (2\cos\theta)\lambda + 1$

Thus the characteristic equation of A is $\lambda^2 - (2\cos\theta)\lambda + 1 = 0$. For this equation $b^2 - 4ac = 4\cos^2\theta - 4 = 4(\cos^2\theta - 1)$. Since $0 < \theta < \pi$, $\cos^2\theta < 1$ hence $b^2 - 4ac < 0$, so the matrix has no real eigenvalues or eigenvectors.

(b) The matrix rotates vectors in R^2 through the angle θ. Since $0 < \theta < \pi$, no vector is transformed into a vector with the same or opposite direction.

3. (a) If $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$ then

$$S = \begin{bmatrix} \sqrt{d_1} & 0 & \cdots & 0 \\ 0 & \sqrt{d_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sqrt{d_n} \end{bmatrix}.$$

(b) If A is a diagonalizable matrix with nonnegative eigenvalues, there is a matrix P such that $P^{-1}AP = D$ and $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ is a diagonal matrix with nonnegative entries on the main diagonal. By part (a) there is a matrix S_1 such that $S_1^2 = D$. Let $S = PS_1P^{-1}$, then

$$\begin{aligned} S^2 &= (PS_1P^{-1})(PS_1P^{-1}) \\ &= PS_1^2P^{-1} \\ &= PDP^{-1} \\ &= A. \end{aligned}$$

(c) A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 9$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Thus $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ diagonalizes A, and

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

If $S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then $S_1^2 = D$.

$$\begin{aligned} S &= PS_1P^{-1} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \end{aligned}$$

$S^2 = A$, as required.

9. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -6 \\ -1 & \lambda - 2 \end{vmatrix} = \lambda^2 - 5\lambda$

The characteristic equation of A is $\lambda^2 - 5\lambda = 0$, hence $A^2 - 5A = 0$ or $A^2 = 5A$.

$$A^2 = 5A = 5\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix}$$

$$A^3 = 5A^2 = 5\begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix} = \begin{bmatrix} 75 & 150 \\ 25 & 50 \end{bmatrix}$$

$$A^4 = 5A^3 = 5\begin{bmatrix} 75 & 150 \\ 25 & 50 \end{bmatrix} = \begin{bmatrix} 375 & 750 \\ 125 & 250 \end{bmatrix}$$

$$A^5 = 5A^4 = 5\begin{bmatrix} 375 & 750 \\ 125 & 250 \end{bmatrix} = \begin{bmatrix} 1875 & 3750 \\ 625 & 1250 \end{bmatrix}$$

11. Suppose that $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Since $A\mathbf{x} = \begin{bmatrix} c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \vdots \\ c_1x_1 + c_2x_2 + \cdots + c_nx_n \end{bmatrix}$ then either $x_1 = x_2 = \cdots = x_n$, $\lambda = c_1 + c_2 + \cdots + c_n = \operatorname{tr}(A)$ or not all the x_i are equal and λ must be 0.

13. If A is a $k \times k$ nilpotent matrix where $A^n = 0,$ then $\det(\lambda I - A^n) = \det(\lambda I) = \lambda^k$ and A^n has characteristic polynomial $\lambda^k = 0,$ so the eigenvalues of A^n are all 0. But if λ_1 is an eigenvalue of A, then λ_1^n is an eigenvalue of $A^n,$ so all the eigenvalues of A must also be 0.

15. The matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ will diagonalize A and $P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$ thus

$$\begin{aligned} A &= PDP^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \end{aligned}$$

17. If λ is an eigenvalue of A, the λ^3 will be an eigenvalue of A^3 corresponding to the same eigenvector, i.e., if $A\mathbf{x}_1 = \lambda\mathbf{x}_1,$ then $A^3\mathbf{x}_1 = \lambda^3\mathbf{x}_1.$ Since $A^3 = A,$ the eigenvalues must satisfy $\lambda^3 = \lambda$ and the only possibilities are -1, 0, and 1.

Chapter 6

Inner Product Spaces

Section 6.1

Exercise Set 6.1

1. **(a)** $\langle \mathbf{u}, \mathbf{v}\rangle = 1\cdot 3 + 1\cdot 2 = 5$

(b) $\langle k\mathbf{v}, \mathbf{w}\rangle = \langle 3\mathbf{v}, \mathbf{w}\rangle = 9\cdot 0 + 6\cdot(-1) = -6$

(c) $$\begin{aligned}\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle &= \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle \\ &= 1\cdot 0 + 1\cdot(-1) + 3\cdot 0 + 2\cdot(-1) \\ &= -3\end{aligned}$$

(d) $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle} = \sqrt{3^2+2^2} = \sqrt{13}$

(e) $$\begin{aligned}d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\| \\ &= \|(-2, -1)\| \\ &= \sqrt{(-2)^2+(-1)^2} \\ &= \sqrt{5}\end{aligned}$$

(f) $$\begin{aligned}\|\mathbf{u}-k\mathbf{v}\| &= \|(1, 1)-3(3, 2)\| \\ &= \|(-8, -5)\| \\ &= \sqrt{(-8)^2+(-5)^2} \\ &= \sqrt{89}\end{aligned}$$

3. **(a)** $\langle \mathbf{u}, \mathbf{v}\rangle = 3(4)+(-2)(5) = 2$

$\langle \mathbf{v}, \mathbf{u}\rangle = 4(3)+5(-2) = 2$

(b) $$\begin{aligned}\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle &= \langle (7, 3), (-1, 6)\rangle \\ &= 7(-1)+3(6) \\ &= 11\end{aligned}$$

$$\begin{aligned}&\langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle \\ &= (3(-1)+(-2)(6)) + (4(-1)+5(6)) \\ &= -15+26 \\ &= 11\end{aligned}$$

(c) $$\begin{aligned}\langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle &= \langle (3, -2), (3, 11)\rangle \\ &= 3(3)+(-2)(11) \\ &= -13\end{aligned}$$

$$\begin{aligned}&\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle \\ &= (3(4)+(-2)(5)) + (3(-1)+(-2)(6)) \\ &= 2-15 \\ &= -13\end{aligned}$$

(d) $$\begin{aligned}\langle k\mathbf{u}, \mathbf{v}\rangle &= \langle (-12, 8), (4, 5)\rangle \\ &= -12(4)+8(5) \\ &= -8\end{aligned}$$

$k\langle \mathbf{u}, \mathbf{v}\rangle = -4(2) = -8$

$$\begin{aligned}\langle \mathbf{u}, k\mathbf{v}\rangle &= \langle (3, -2), (-16, -20)\rangle \\ &= 3(-16)+(-2)(-20) \\ &= -8\end{aligned}$$

(e) $\langle \mathbf{0}, \mathbf{v}\rangle = \langle (0, 0), (4, 5)\rangle = 0(4)+0(5) = 0$

$\langle \mathbf{v}, \mathbf{0}\rangle = \langle (4, 5), (0, 0)\rangle = 4(0)+5(0) = 0$

5. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1\end{bmatrix}$. Then $A^T = \begin{bmatrix} 2 & 1 \\ 1 & 1\end{bmatrix} = A$ so $A^TA = A^2 = \begin{bmatrix} 5 & 3 \\ 3 & 2\end{bmatrix}$.

(a) $$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle &= \mathbf{v}^TA^TA\mathbf{u} \\ &= [-1 \quad 1]\begin{bmatrix} 5 & 3 \\ 3 & 2\end{bmatrix}\begin{bmatrix} 2 \\ 1\end{bmatrix} \\ &= [-2 \quad -1]\begin{bmatrix} 2 \\ 1\end{bmatrix} \\ &= -4-1 \\ &= -5\end{aligned}$$

(b) $$\begin{aligned}\langle \mathbf{v}, \mathbf{w}\rangle &= \mathbf{w}^TA^TA\mathbf{v} \\ &= [0 \quad -1]\begin{bmatrix} 5 & 3 \\ 3 & 2\end{bmatrix}\begin{bmatrix} -1 \\ 1\end{bmatrix} \\ &= [-3 \quad -2]\begin{bmatrix} -1 \\ 1\end{bmatrix} \\ &= 3-2 \\ &= 1\end{aligned}$$

(c) $$\begin{aligned}\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle &= \mathbf{w}^TA^TA(\mathbf{u}+\mathbf{v}) \\ &= [0 \quad -1]\begin{bmatrix} 5 & 3 \\ 3 & 2\end{bmatrix}\begin{bmatrix} 1 \\ 2\end{bmatrix} \\ &= [-3 \quad -2]\begin{bmatrix} 1 \\ 2\end{bmatrix} \\ &= -3-4 \\ &= -7\end{aligned}$$

(d) $\langle \mathbf{v}, \mathbf{v}\rangle = \mathbf{v}^T A^T A\mathbf{v}$
$$= [-1 \quad 1]\begin{bmatrix}5 & 3\\3 & 2\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = [-2 \quad -1]\begin{bmatrix}-1\\1\end{bmatrix} = 2-1 = 1$$
$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle} = \sqrt{1} = 1$

(e) $\mathbf{v}-\mathbf{w} = \begin{bmatrix}-1\\1\end{bmatrix} - \begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$
$$\langle \mathbf{v}-\mathbf{w}, \mathbf{v}-\mathbf{w}\rangle = (\mathbf{v}-\mathbf{w})^T A^T A(\mathbf{v}-\mathbf{w}) = [-1 \quad 2]\begin{bmatrix}5 & 3\\3 & 2\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = [1 \quad 1]\begin{bmatrix}-1\\2\end{bmatrix} = -1+2 = 1$$
$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v}-\mathbf{w}\| = \sqrt{\langle \mathbf{v}-\mathbf{w}, \mathbf{v}-\mathbf{w}\rangle} = \sqrt{1} = 1$$

(f) $\|\mathbf{v}-\mathbf{w}\|^2 = (d(\mathbf{v}, \mathbf{w}))^2 = 1^2 = 1$

7. (a) $\mathbf{u}^T\mathbf{v} = \begin{bmatrix}3 & 4\\-2 & 8\end{bmatrix}\begin{bmatrix}-1 & 3\\1 & 1\end{bmatrix} = \begin{bmatrix}1 & 13\\10 & 2\end{bmatrix}$

$\langle \mathbf{u}, \mathbf{v}\rangle = \operatorname{tr}(\mathbf{u}^T\mathbf{v}) = 1+2 = 3$

(b) $\mathbf{u}^T\mathbf{v} = \begin{bmatrix}1 & -3\\2 & 5\end{bmatrix}\begin{bmatrix}4 & 6\\0 & 8\end{bmatrix} = \begin{bmatrix}4 & -18\\8 & 52\end{bmatrix}$

$\langle \mathbf{u}, \mathbf{v}\rangle = \operatorname{tr}(\mathbf{u}^T\mathbf{v}) = 4+52 = 56$

9. (a) $A^T = \begin{bmatrix}3 & 0\\0 & 2\end{bmatrix} = A,$ so $A^T A = A^2 = \begin{bmatrix}9 & 0\\0 & 4\end{bmatrix}.$

For $\mathbf{u} = \begin{bmatrix}u_1\\u_2\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix},$
$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{v}^T A^T A\mathbf{u} = [v_1 \quad v_2]\begin{bmatrix}9 & 0\\0 & 4\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = [9v_1 \quad 4v_2]\begin{bmatrix}u_1\\u_2\end{bmatrix} = 9u_1v_1 + 4u_2v_2$$

(b) $\langle \mathbf{u}, \mathbf{v}\rangle = 9(-3)(1) + 4(2)(7) = -27+56 = 29$

11. (a) The matrix is $A = \begin{bmatrix}\sqrt{3} & 0\\0 & \sqrt{5}\end{bmatrix},$ since $A^T A = A^2 = \begin{bmatrix}3 & 0\\0 & 5\end{bmatrix}.$

(b) The matrix is $A = \begin{bmatrix}2 & 0\\0 & \sqrt{6}\end{bmatrix},$ since $A^T A = A^2 = \begin{bmatrix}4 & 0\\0 & 6\end{bmatrix}.$

13. (a) $\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{(-2)^2+5^2+3^2+6^2} = \sqrt{74}$

(b) $\|A\| = 0$ by inspection.

15. (a) $A-B = \begin{bmatrix}6 & -1\\8 & -2\end{bmatrix}$
$$d(A, B) = \|A-B\| = \sqrt{\langle A-B, A-B\rangle} = \sqrt{6^2+(-1)^2+8^2+(-2)^2} = \sqrt{105}$$

(b) $A-B = \begin{bmatrix}3 & 3\\-5 & -2\end{bmatrix}$
$$d(A, B) = \|A-B\| = \sqrt{\langle A-B, A-B\rangle} = \sqrt{3^2+3^2+(-5)^2+(-2)^2} = \sqrt{47}$$

17. $\langle \mathbf{p}, \mathbf{q}\rangle$
$$= p(-1)q(-1) + p(0)q(0) + p(1)q(1) + p(2)q(2) = (-2)(2)+(0)(1)+(2)(2)+(10)(5) = -4+0+4+50 = 50$$
$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p}\rangle} = \sqrt{[p(-1)]^2+[p(0)]^2+[p(1)]^2+[p(2)]^2} = \sqrt{(-2)^2+0^2+2^2+10^2} = \sqrt{108} = 6\sqrt{3}$$

19. $\mathbf{u}-\mathbf{v}=(-3, -3)$

(a) $$\begin{aligned} d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\| \\ &= \sqrt{\langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle} \\ &= \sqrt{(-3)^2+(-3)^2} \\ &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

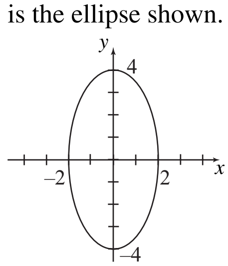
(b) $$\begin{aligned} d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\| \\ &= \sqrt{\langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle} \\ &= \sqrt{3(-3)^2+2(-3)^2} \\ &= \sqrt{45} \\ &= 3\sqrt{5} \end{aligned}$$

(c) $$A^TA=\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix}=\begin{bmatrix}2 & -1\\ -1 & 13\end{bmatrix}$$

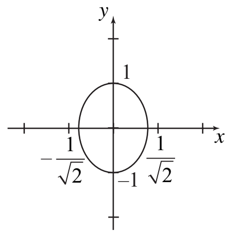
$$\begin{aligned} \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle &= (\mathbf{u}-\mathbf{v})^T A^T A(\mathbf{u}-\mathbf{v}) \\ &= [-3 \quad -3]\begin{bmatrix}2 & -1\\ -1 & 13\end{bmatrix}\begin{bmatrix}-3\\ -3\end{bmatrix} \\ &= [-3 \quad -36]\begin{bmatrix}-3\\ -3\end{bmatrix} \\ &= 9+108 \\ &= 117 \end{aligned}$$

$$\begin{aligned} d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\| \\ &= \sqrt{\langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle} \\ &= \sqrt{117} \\ &= 3\sqrt{13} \end{aligned}$$

21. (a) For $\mathbf{u}=(x, y)$, $\|\mathbf{u}\|=\langle \mathbf{u}, \mathbf{u}\rangle^{1/2}=\sqrt{\frac{1}{4}x^2+\frac{1}{16}y^2}$ and the unit circle is $\sqrt{\frac{1}{4}x^2+\frac{1}{16}y^2}=1$ or $\frac{x^2}{4}+\frac{y^2}{16}=1$ which is the ellipse shown.



(b) For $\mathbf{u} = (x, y)$, $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u}\rangle^{1/2} = \sqrt{2x^2 + y^2}$ and the unit circle is $\sqrt{2x^2 + y^2} = 1$ or $\frac{x^2}{\frac{1}{2}} + \frac{y^2}{1} = 1$ which is the ellipse shown.



25.
$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle + \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle \\ &= \langle \mathbf{u}, \mathbf{u}+\mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}+\mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{u}-\mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}-\mathbf{v}\rangle \\ &= \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle \\ &= 2\langle \mathbf{u}, \mathbf{u}\rangle + 2\langle \mathbf{v}, \mathbf{v}\rangle \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2\end{aligned}$$

27. Let $V = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then

$$\begin{aligned}\langle V, V\rangle &= 0(0) + (1)(-1) + (-1)(1) + (0)(0) \\ &= -2 < 0\end{aligned}$$

so Axiom 4 fails.

29. **(a)**
$$\begin{aligned}\langle \mathbf{p}, \mathbf{q}\rangle &= \int_{-1}^{1} (1 - x + x^2 + 5x^3)(x - 3x^2)\,dx \\ &= \int_{-1}^{1} (x - 4x^2 + 4x^3 + 2x^4 - 15x^5)\,dx \\ &= \left(\frac{x^2}{2} - \frac{4x^3}{3} + x^4 + \frac{2x^5}{5} - \frac{5x^6}{2}\right)\Bigg|_{-1}^{1} \\ &= -\frac{29}{15} - \left(-\frac{1}{15}\right) \\ &= -\frac{28}{15}\end{aligned}$$

(b)
$$\begin{aligned}\langle \mathbf{p}, \mathbf{q}\rangle &= \int_{-1}^{1} (x - 5x^3)(2 + 8x^2)\,dx \\ &= \int_{-1}^{1} (2x - 2x^3 - 40x^5)\,dx \\ &= \left(x^2 - \frac{x^4}{2} - \frac{20x^6}{3}\right)\Bigg|_{-1}^{1} \\ &= -\frac{37}{6} - \left(-\frac{37}{6}\right) \\ &= 0\end{aligned}$$

True/False 6.1

(a) True; with $w_1 = w_2 = 1$, the weighted Euclidean inner product is the dot product on R^2.

(b) False; for example, using the dot product on R^2 with $\mathbf{u} = (0, -1)$ and $\mathbf{v} = (0, 5)$, $\langle \mathbf{u}, \mathbf{v} \rangle = 0(0) - 1(5) = -5$.

(c) True; first apply Axiom 1, then Axiom 2.

(d) True; applying Axiom 3, Axiom 1, then Axiom 3 and Axiom 1 again gives
$$\begin{aligned}\langle k\mathbf{u}, k\mathbf{v} \rangle &= k\langle \mathbf{u}, k\mathbf{v} \rangle \\ &= k\langle k\mathbf{v}, \mathbf{u} \rangle \\ &= k(k)\langle \mathbf{v}, \mathbf{u} \rangle \\ &= k^2\langle \mathbf{u}, \mathbf{v} \rangle.\end{aligned}$$

(e) False; for example, using the dot product on R^2 with $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (0, 1)$, $\langle \mathbf{u}, \mathbf{v} \rangle = 1(0) + 0(1) = 0$, i.e., $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(f) True; if $0 = \|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle$, then $\mathbf{v} = \mathbf{0}$ by Axiom 4.

(g) False; the matrix A must be invertible, otherwise there is a nontrivial solution $\mathbf{x}_0$ to the system $A\mathbf{x} = \mathbf{0}$ and $\langle \mathbf{x}_0, \mathbf{x}_0 \rangle = A\mathbf{x}_0 \cdot A\mathbf{x}_0 = \mathbf{0} \cdot \mathbf{0} = 0$, but $\mathbf{x}_0 \neq \mathbf{0}$.

Section 6.2

Exercise Set 6.2

1. (a) $\|\mathbf{u}\| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$

$\|\mathbf{v}\| = \sqrt{2^2 + 4^2} = \sqrt{20}$

$\langle \mathbf{u}, \mathbf{v} \rangle = 1(2) + (-3)(4) = -10$

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-10}{\sqrt{10}\sqrt{20}} = -\frac{10}{\sqrt{200}} = -\frac{1}{\sqrt{2}}$$

(b) $\|\mathbf{u}\| = \sqrt{(-1)^2 + 0^2} = 1$

$\|\mathbf{v}\| = \sqrt{3^2 + 8^2} = \sqrt{73}$

$\langle \mathbf{u}, \mathbf{v} \rangle = (-1)(3) + 0(8) = -3$

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-3}{\sqrt{73}}$$

(c) $\|\mathbf{u}\| = \sqrt{(-1)^2 + 5^2 + 2^2} = \sqrt{30}$

$\|\mathbf{v}\| = \sqrt{2^2 + 4^2 + (-9)^2} = \sqrt{101}$

$\langle \mathbf{u}, \mathbf{v} \rangle = (-1)(2) + 5(4) + 2(-9) = 0$

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{0}{\sqrt{30}\sqrt{101}} = 0$$

(d) $\|\mathbf{u}\| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$

$\|\mathbf{v}\| = \sqrt{1^2 + 0^2 + (-3)^2} = \sqrt{10}$

$\langle \mathbf{u}, \mathbf{v} \rangle = 4(1) + 1(0) + 8(-3) = -20$

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = -\frac{20}{9\sqrt{10}}$$

(e) $\|\mathbf{u}\| = \sqrt{1^2 + 0^2 + 1^2 + 0^2} = \sqrt{2}$

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{(-3)^2 + (-3)^2 + (-3)^2 + (-3)^2} \\ &= \sqrt{36} \\ &= 6\end{aligned}$$

$\langle \mathbf{u}, \mathbf{v} \rangle = 1(-3) + 0(-3) + 1(-3) + 0(-3) = -6$

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-6}{6\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

(f) $\|\mathbf{u}\| = \sqrt{2^2 + 1^2 + 7^2 + (-1)^2} = \sqrt{55}$

$\|\mathbf{v}\| = \sqrt{4^2 + 0^2 + 0^2 + 0^2} = 4$

$\langle \mathbf{u}, \mathbf{v} \rangle = 2(4) + 1(0) + 7(0) + (-1)(0) = 8$

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{8}{4\sqrt{55}} = \frac{2}{\sqrt{55}}$$

3. (a)
$$\begin{aligned}\|A\| &= \sqrt{\langle A, A \rangle} \\ &= \sqrt{2^2 + 6^2 + 1^2 + (-3)^2} \\ &= \sqrt{50} \\ &= 5\sqrt{2}\end{aligned}$$

$\|B\| = \sqrt{\langle B, B \rangle} = \sqrt{3^2 + 2^2 + 1^2 + 0^2} = \sqrt{14}$

$\langle A, B \rangle = 2(3) + 6(2) + 1(1) + (-3)(0) = 19$

$$\cos\theta = \frac{\langle A, B \rangle}{\|A\|\|B\|} = \frac{19}{5\sqrt{2}\sqrt{14}} = \frac{19}{10\sqrt{7}}$$

(b)
$$\begin{aligned}\|A\| &= \sqrt{\langle A, A \rangle} \\ &= \sqrt{2^2 + 4^2 + (-1)^2 + 3^2} \\ &= \sqrt{30}\end{aligned}$$

$$\begin{aligned}\|B\| &= \sqrt{\langle B, B \rangle} \\ &= \sqrt{(-3)^2 + 1^2 + 4^2 + 2^2} \\ &= \sqrt{30}\end{aligned}$$

$\langle A, B\rangle = 2(-3)+4(1)+(-1)(4)+3(2) = 0$

$\cos\theta = \dfrac{\langle A, B\rangle}{\|A\|\|B\|} = \dfrac{0}{30} = 0$

5. $\langle \mathbf{p}, \mathbf{q}\rangle = 1(0)+(-1)(2)+2(1) = 0$

7. $0 = \langle \mathbf{u}, \mathbf{w}\rangle = 2(1)+k(2)+6(3) = 20+2k$

Thus $k = -10$ and $\mathbf{u} = (2, -10, 6)$.

$0 = \langle \mathbf{v}, \mathbf{w}\rangle = l(1)+5(2)+3(3) = l+19$

Thus, $l = -19$ and $\mathbf{v} = (-19, 5, 3)$.

$\langle \mathbf{u}, \mathbf{v}\rangle = 2(-19)-10(5)+6(3) = -70 \neq 0$

Thus, there are no scalars such that the vectors are mutually orthogonal.

9. **(a)** $\langle \mathbf{u}, \mathbf{v}\rangle = 2(1)+1(7)+3k = 9+3k$

$\mathbf{u}$ and $\mathbf{v}$ are orthogonal for $k = -3$.

(b) $\langle \mathbf{u}, \mathbf{v}\rangle = k(k)+k(5)+1(6) = k^2+5k+6$

$k^2+5k+6 = (k+2)(k+3)$

$\mathbf{u}$ and $\mathbf{v}$ are orthogonal for $k = -2, -3$.

11. **(a)** $|\langle \mathbf{u}, \mathbf{v}\rangle| = |3(4)+2(-1)| = |10| = 10$

$$\begin{aligned}\|\mathbf{u}\|\|\mathbf{v}\| &= \sqrt{3^2+2^2}\sqrt{4^2+(-1)^2}\\ &= \sqrt{13}\sqrt{17}\\ &= \sqrt{221}\\ &\approx 14.9\end{aligned}$$

(b) $|\langle \mathbf{u}, \mathbf{v}\rangle| = |-3(2)+1(-1)+0(3)| = |-7| = 7$

$$\begin{aligned}\|\mathbf{u}\|\|\mathbf{v}\| &= \sqrt{(-3)^2+1^2+0^2}\sqrt{2^2+(-1)^2+3^2}\\ &= \sqrt{10}\sqrt{14}\\ &= \sqrt{140}\\ &\approx 11.8\end{aligned}$$

(c) $|\langle \mathbf{u}, \mathbf{v}\rangle| = |-4(8)+2(-4)+1(-2)| = |-42| = 42$

$$\begin{aligned}\|\mathbf{u}\|\|\mathbf{v}\| &= \sqrt{(-4)^2+2^2+1^2}\sqrt{8^2+(-4)^2+(-2)^2}\\ &= \sqrt{21}\sqrt{84}\\ &= \sqrt{1764}\\ &= 42\end{aligned}$$

(d) $|\langle \mathbf{u}, \mathbf{v}\rangle| = |0(-1)+(-2)(-1)+2(1)+1(1)|$
$= |5|$
$= 5$

$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{0^2+(-2)^2+2^2+1^2}\sqrt{(-1)^2+(-1)^2+1^2+1^2}$
$= \sqrt{9}\sqrt{4}$
$= 6$

13. $\langle \mathbf{u}, \mathbf{w}_1\rangle = 0$ by inspection.
$\langle \mathbf{u}, \mathbf{w}_2\rangle = -1(1)+1(-1)+0(3)+2(0) = -2$
Since **u** is not orthogonal to $\mathbf{w}_2$, it is not orthogonal to the subspace.

15. **(a)** $W^\perp$ will have dimension 1. A normal to the plane is **u** = (1, −2, −3), so $W^\perp$ will consist of all scalar multiples of **u** or t**u** = (t, −2t, −3t) so parametric equations for $W^\perp$ are $x = t$, $y = -2t$, $z = -3t$.

(b) The line contains the vector **u** = (2, −5, 4), thus $W^\perp$ is the plane through the origin with normal **u**, or $2x - 5y + 4z = 0$.

(c) The intersection of the planes is the line $x + z = 0$ in the xz plane ($y = 0$) which can be parametrized as $(x, y, z) = (t, 0, -t)$ or $t(1, 0, -1)$. $W^\perp$ is the plane with normal (1, 0, −1) or $x - z = 0$.

17. $\|\mathbf{u}-\mathbf{v}\|^2 = \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle$
$= \langle \mathbf{u}, \mathbf{u}-\mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}-\mathbf{v}\rangle$
$= \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle$
$= \|\mathbf{u}\|^2 - 0 - 0 + \|\mathbf{v}\|^2$
$= 2$
Thus $\|\mathbf{u}-\mathbf{v}\| = \sqrt{2}$.

19. $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\} = k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_r\mathbf{u}_r$ where $k_1, k_2, \ldots, k_r$ are arbitrary scalars. Let $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$.
$\langle \mathbf{w}, \mathbf{v}\rangle = \langle \mathbf{w}, k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_r\mathbf{u}_r\rangle$
$= k_1\langle \mathbf{w}, \mathbf{u}_1\rangle + k_2\langle \mathbf{w}, \mathbf{u}_2\rangle + \cdots + k_r\langle \mathbf{w}, \mathbf{u}_r\rangle$
$= 0 + 0 + \cdots + 0$
$= 0$
Thus if **w** is orthogonal to each vector $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$, then **w** must be orthogonal to every vector in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$.

21. Suppose that **v** is orthogonal to every basis vector. Then, as in exercise 19, **v** is orthogonal to the span of the set of basis vectors, which is all of W, hence **v** is in $W^\perp$. If **v** is not orthogonal to every basis vector, then **v** clearly cannot be in $W^\perp$. Thus $W^\perp$ consists of all vectors orthogonal to every basis vector.

27. Using the figure in the text, $\overrightarrow{AB} = \mathbf{v} + \mathbf{u}$ and $\overrightarrow{BC} = \mathbf{v} - \mathbf{u}$. Use the Euclidean inner product on R^2 and note that $\|\mathbf{u}\| = \|\mathbf{v}\|$.

$$\begin{aligned}\left\langle \overrightarrow{AB}, \overrightarrow{BC} \right\rangle &= \langle \mathbf{v}+\mathbf{u}, \mathbf{v}-\mathbf{u}\rangle \\ &= \langle \mathbf{v}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{u}, \mathbf{u}\rangle \\ &= \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \\ &= 0\end{aligned}$$

Thus $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are orthogonal, so the angle at B is a right angle.

31. **(a)** $W^\perp$ is the line through the origin which is perpendicular to $y = x$. Thus $W^\perp$ is the line $y = -x$.

(b) $W^\perp$ is the plane through the origin which is perpendicular to the y-axis. Thus, $W^\perp$ is the xz-plane.

(c) $W^\perp$ is the line through the origin which is perpendicular to the yz-plane. Thus $W^\perp$ is the x-axis.

True/False 6.2

(a) False; if $\mathbf{u}$ is orthogonal to every vector of a subspace W, then $\mathbf{u}$ is in $W^\perp$.

(b) True; $W \cap W^\perp = \{\mathbf{0}\}$.

(c) True; for any vector $\mathbf{w}$ in W, $\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle = 0$, so $\mathbf{u} + \mathbf{v}$ is in $W^\perp$.

(d) True; for any vector $\mathbf{w}$ in W, $\langle k\mathbf{u}, \mathbf{w}\rangle = k\langle \mathbf{u}, \mathbf{w}\rangle = k(0) = 0$, so $k\mathbf{u}$ is in $W^\perp$.

(e) False; if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal $|\langle \mathbf{u}, \mathbf{v}\rangle| = |0| = 0$.

(f) False; if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ thus $\|\mathbf{u}+\mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2} \neq \|\mathbf{u}\| + \|\mathbf{v}\|$

Section 6.3

Exercise Set 6.3

1. **(a)** $\langle (0, 1), (2, 0)\rangle = 0 + 0 = 0$
The set is orthogonal.

(b) $\left\langle \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\rangle = -\frac{1}{2} + \frac{1}{2} = 0$
The set is orthogonal.

(c) $$\begin{aligned}\left\langle \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\rangle &= -\frac{1}{2} - \frac{1}{2} \\ &= -1 \\ &\neq 0\end{aligned}$$
The set is not orthogonal.

(d) $\langle (0, 0), (0, 1)\rangle = 0 + 0 = 0$
The set is orthogonal.

3. **(a)** $$\begin{aligned}&\left\langle \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\right\rangle \\ &= \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} \\ &= 0\end{aligned}$$
$$\begin{aligned}&\left\langle \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right\rangle \\ &= -\frac{1}{2} + 0 + \frac{1}{2} \\ &= 0\end{aligned}$$
$$\begin{aligned}&\left\langle \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right\rangle \\ &= -\frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} \\ &= -\frac{2}{\sqrt{6}} \neq 0\end{aligned}$$
The set is not orthogonal.

(b) $$\begin{aligned}\left\langle \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\right\rangle &= \frac{4}{9} - \frac{2}{9} - \frac{2}{9} \\ &= 0\end{aligned}$$
$$\left\langle \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\right\rangle = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$$
$$\left\langle \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\right\rangle = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$$
The set is orthogonal.

(c) $\left\langle (1, 0, 0), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\rangle = 0+0+0=0$

$\langle (1, 0, 0), (0, 0, 1)\rangle = 0+0+0=0$

$$\left\langle \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), (0, 0, 1)\right\rangle = 0+0+\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \neq 0$$

The set is not orthogonal.

(d) $\left\langle \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)\right\rangle$

$$= \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{12}} + 0 = 0$$

The set is orthogonal.

5. (a) $\|p_1(x)\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = \sqrt{1} = 1$

$$\|p_2(x)\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}} = \sqrt{1} = 1$$

$$\|p_3(x)\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}} = \sqrt{1} = 1$$

$$\langle p_1(x), p_2(x)\rangle = \frac{2}{3}\left(\frac{2}{3}\right) - \frac{2}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(-\frac{2}{3}\right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$$

$$\langle p_1(x), p_3(x)\rangle = \frac{2}{3}\left(\frac{1}{3}\right) - \frac{2}{3}\left(\frac{2}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$$

$$\langle p_2(x), p_3(x)\rangle = \frac{2}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right) - \frac{2}{3}\left(\frac{2}{3}\right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$$

The set is an orthonormal set.

(b) $\|p_1(x)\| = \sqrt{1^2} = 1$

$$\|p_2(x)\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{1} = 1$$

$\|p_3(x)\| = \sqrt{1^2} = 1$

$\langle p_1(x), p_2(x)\rangle = 1(0) + 0(1) + 0(1) = 0$

$\langle p_1(x), p_3(x)\rangle = 1(0) + 0(0) + 0(1) = 0$

$$\langle p_2(x), p_3(x)\rangle = 0(0) + \frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}} \neq 0$$

The set is not an orthonormal set.

7. (a) $\langle (-1, 2), (6, 3)\rangle = -6+6=0$

$\|(-1, 2)\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

$$\frac{(-1, 2)}{\|(-1, 2)\|} = \frac{(-1, 2)}{\sqrt{5}} = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$\|(6, 3)\| = \sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$

$$\frac{(6, 3)}{\|(6, 3)\|} = \frac{(6, 3)}{3\sqrt{5}} = \left(\frac{6}{3\sqrt{5}}, \frac{3}{3\sqrt{5}}\right) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

The vectors $\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ are an orthonormal set.

(b) $\langle (1, 0, -1), (2, 0, 2)\rangle = 2+0-2=0$

$\langle (1, 0, -1), (0, 5, 0)\rangle = 0+0+0=0$

$\langle (2, 0, 2), (0, 5, 0)\rangle = 0+0+0=0$

$\|(1, 0, -1)\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$

$$\frac{(1, 0, -1)}{\|(1, 0, -1)\|} = \frac{(1, 0, -1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

$$\|(2, 0, 2)\| = \sqrt{2^2 + 0^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\begin{aligned}\frac{(2, 0, 2)}{\|(2, 0, 2)\|} &= \frac{(2, 0, 2)}{2\sqrt{2}} \\ &= \left(\frac{2}{2\sqrt{2}}, 0, \frac{2}{2\sqrt{2}}\right) \\ &= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\end{aligned}$$

$$\|(0, 5, 0)\| = \sqrt{0^2 + 5^2 + 0^2} = 5$$

$$\frac{(0, 5, 0)}{\|(0, 5, 0)\|} = \frac{(0, 5, 0)}{5} = (0, 1, 0)$$

The vectors $\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, and (0, 1, 0) are an orthonormal set.

(c) $$\begin{aligned}\left\langle\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right), \left(-\frac{1}{2}, \frac{1}{2}, 0\right)\right\rangle &= -\frac{1}{10} + \frac{1}{10} + 0 \\ &= 0\end{aligned}$$

$$\begin{aligned}\left\langle\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right), \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)\right\rangle &= \frac{1}{15} + \frac{1}{15} - \frac{2}{15} \\ &= 0\end{aligned}$$

$$\begin{aligned}\left\langle\left(-\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)\right\rangle &= -\frac{1}{6} + \frac{1}{6} + 0 \\ &= 0\end{aligned}$$

$$\begin{aligned}\left\|\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)\right\| &= \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^2} \\ &= \sqrt{\frac{3}{25}} \\ &= \frac{\sqrt{3}}{5}\end{aligned}$$

$$\begin{aligned}\frac{\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)}{\left\|\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)\right\|} &= \frac{5}{\sqrt{3}}\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) \\ &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\end{aligned}$$

$$\begin{aligned}\left\|\left(-\frac{1}{2}, \frac{1}{2}, 0\right)\right\| &= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 0} \\ &= \sqrt{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\frac{\left(-\frac{1}{2}, \frac{1}{2}, 0\right)}{\left\|\left(-\frac{1}{2}, \frac{1}{2}, 0\right)\right\|} &= \frac{\sqrt{2}}{1}\left(-\frac{1}{2}, \frac{1}{2}, 0\right) \\ &= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\end{aligned}$$

$$\begin{aligned}\left\|\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)\right\| &= \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} \\ &= \sqrt{\frac{6}{9}} \\ &= \frac{\sqrt{6}}{3}\end{aligned}$$

$$\begin{aligned}\frac{\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)}{\left\|\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)\right\|} &= \frac{3}{\sqrt{6}}\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right) \\ &= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)\end{aligned}$$

The vectors $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$ are an orthonormal set.

9. $$\begin{aligned}\|\mathbf{v}_1\| &= \sqrt{\left(-\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + 0^2} \\ &= \sqrt{\frac{9}{25} + \frac{16}{25}} \\ &= \sqrt{1} \\ &= 1\end{aligned}$$

$$\begin{aligned}\|\mathbf{v}_2\| &= \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + 0^2} \\ &= \sqrt{\frac{16}{25} + \frac{9}{25}} \\ &= \sqrt{1} \\ &= 1\end{aligned}$$

$$\|\mathbf{v}_3\| = \sqrt{0^2 + 0^2 + 1^2} = 1$$

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = -\frac{12}{25} + \frac{12}{25} + 0 = 0$$

$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0+0+0=0$
$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0+0+0=0$

(a) $\mathbf{u} = (1, -1, 2)$

$\langle \mathbf{u}, \mathbf{v}_1 \rangle = -\frac{3}{5}-\frac{4}{5}+0=-\frac{7}{5}$

$\langle \mathbf{u}, \mathbf{v}_2 \rangle = \frac{4}{5}-\frac{3}{5}+0=\frac{1}{5}$

$\langle \mathbf{u}, \mathbf{v}_3 \rangle = 0+0+2=2$

Thus, $(1, -1, 2) = -\frac{7}{5}\mathbf{v}_1+\frac{1}{5}\mathbf{v}_2+2\mathbf{v}_3.$

(b) $\mathbf{u} = (3, -7, 4)$

$\langle \mathbf{u}, \mathbf{v}_1 \rangle = -\frac{9}{5}-\frac{28}{5}+0=-\frac{37}{5}$

$\langle \mathbf{u}, \mathbf{v}_2 \rangle = \frac{12}{5}-\frac{21}{5}+0=-\frac{9}{5}$

$\langle \mathbf{u}, \mathbf{v}_3 \rangle = 0+0+4=4$

Thus, $(3, -7, 4) = -\frac{37}{5}\mathbf{v}_1-\frac{9}{5}\mathbf{v}_2+4\mathbf{v}_3.$

(c) $\mathbf{u} = \left(\frac{1}{7}, -\frac{3}{7}, \frac{5}{7}\right)$

$\langle \mathbf{u}, \mathbf{v}_1 \rangle = -\frac{3}{35}-\frac{12}{35}+0=-\frac{15}{35}=-\frac{3}{7}$

$\langle \mathbf{u}, \mathbf{v}_2 \rangle = \frac{4}{35}-\frac{9}{35}+0=-\frac{5}{35}=-\frac{1}{7}$

$\langle \mathbf{u}, \mathbf{v}_3 \rangle = 0+0+\frac{5}{7}=\frac{5}{7}$

Thus, $\left(\frac{1}{7}, -\frac{3}{7}, \frac{5}{7}\right) = -\frac{3}{7}\mathbf{v}_1-\frac{1}{7}\mathbf{v}_2+\frac{5}{7}\mathbf{v}_3.$

11. (a) $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 2-2-12+12=0$
$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = -3-8+3+8=0$
$\langle \mathbf{v}_1, \mathbf{v}_4 \rangle = 4-6+6-4=0$
$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = -6+4-4+6=0$
$\langle \mathbf{v}_2, \mathbf{v}_4 \rangle = 8+3-8-3=0$
$\langle \mathbf{v}_3, \mathbf{v}_4 \rangle = -12+12+2-2=0$

Thus, since $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a set of nonzero orthogonal vectors in R^4, S is linearly independent. Since S contains 4 linearly independent orthogonal vectors in R^4, it must be an orthogonal basis for R^4.

(b) $\|\mathbf{v}_1\|^2 = 1^2+(-2)^2+3^2+(-4)^2=30$
$\|\mathbf{v}_2\|^2 = 2^2+1^2+(-4)^2+(-3)^2=30$
$\|\mathbf{v}_3\|^2 = (-3)^2+4^2+1^2+(-2)^2=30$
$\|\mathbf{v}_4\|^2 = 4^2+3^2+2^2+1^2=30$
$\langle \mathbf{u}, \mathbf{v}_1 \rangle = -1-4+9-28=-24$
$\langle \mathbf{u}, \mathbf{v}_2 \rangle = -2+2-12-21=-33$
$\langle \mathbf{u}, \mathbf{v}_3 \rangle = 3+8+3-14=0$
$\langle \mathbf{u}, \mathbf{v}_4 \rangle = -4+6+6+7=15$

$$\mathbf{u} = \frac{-24}{30}\mathbf{v}_1+\frac{-33}{30}\mathbf{v}_2+\frac{0}{30}\mathbf{v}_3+\frac{15}{30}\mathbf{v}_4$$
$$= -\frac{4}{5}\mathbf{v}_1-\frac{11}{10}\mathbf{v}_2+0\mathbf{v}_3+\frac{1}{2}\mathbf{v}_4$$

13. (a) $\langle \mathbf{w}, \mathbf{u}_1 \rangle = \frac{4}{3}+0+\frac{10}{3}=\frac{14}{3}$

$\langle \mathbf{w}, \mathbf{u}_2 \rangle = \frac{2}{3}+0-\frac{10}{3}=-\frac{8}{3}$

$\langle \mathbf{w}, \mathbf{u}_3 \rangle = \frac{4}{3}+0-\frac{5}{3}=-\frac{1}{3}$

Thus, $\mathbf{w} = \frac{14}{3}\mathbf{u}_1-\frac{8}{3}\mathbf{u}_2-\frac{1}{3}\mathbf{u}_3.$

(b) $\langle \mathbf{w}, \mathbf{u}_1 \rangle = -\frac{3}{\sqrt{11}}+\frac{1}{\sqrt{11}}+\frac{2}{\sqrt{11}}=0$

$\langle \mathbf{w}, \mathbf{u}_2 \rangle = \frac{1}{\sqrt{6}}+\frac{2}{\sqrt{6}}+\frac{2}{\sqrt{6}}=\frac{5}{\sqrt{6}}$

$\langle \mathbf{w}, \mathbf{u}_3 \rangle = \frac{1}{\sqrt{66}}-\frac{4}{\sqrt{66}}+\frac{14}{\sqrt{66}}=\frac{11}{\sqrt{66}}$

Thus, $\mathbf{w} = 0\mathbf{u}_1+\frac{5}{\sqrt{6}}\mathbf{u}_2+\frac{11}{\sqrt{66}}\mathbf{u}_3.$

15. (a) $\|\mathbf{v}_1\|^2 = 1^2 + 1^2 + 1^2 + 1^2 = 4$

$\|\mathbf{v}_2\|^2 = 1^2 + 1^2 + (-1)^2 + (-1)^2 = 4$

$\|\mathbf{v}_3\|^2 = 1^2 + (-1)^2 + 1^2 + (-1)^2 = 4$

$\langle \mathbf{x}, \mathbf{v}_1 \rangle = 1 + 2 + 0 - 2 = 1$

$\langle \mathbf{x}, \mathbf{v}_2 \rangle = 1 + 2 + 0 + 2 = 5$

$\langle \mathbf{x}, \mathbf{v}_3 \rangle = 1 - 2 + 0 + 2 = 1$

$$\begin{aligned}\text{proj}_W \mathbf{x} &= \frac{1}{4}\mathbf{v}_1 + \frac{5}{4}\mathbf{v}_2 + \frac{1}{4}\mathbf{v}_3 \\ &= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) + \left(\frac{5}{4}, \frac{5}{4}, -\frac{5}{4}, -\frac{5}{4}\right) + \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right) \\ &= \left(\frac{7}{4}, \frac{5}{4}, -\frac{3}{4}, -\frac{5}{4}\right)\end{aligned}$$

(b) $\|\mathbf{v}_1\|^2 = 0^2 + 1^2 + (-4)^2 + (-1)^2 = 18$

$\|\mathbf{v}_2\|^2 = 3^2 + 5^2 + 1^2 + 1^2 = 36$

$\|\mathbf{v}_3\|^2 = 1^2 + 0^2 + 1^2 + (-4)^2 = 18$

$\langle \mathbf{x}, \mathbf{v}_1 \rangle = 0 + 2 + 0 + 2 = 4$

$\langle \mathbf{x}, \mathbf{v}_2 \rangle = 3 + 10 + 0 - 2 = 11$

$\langle \mathbf{x}, \mathbf{v}_3 \rangle = 1 + 0 + 0 + 8 = 9$

$$\begin{aligned}\text{proj}_W \mathbf{x} &= \frac{4}{18}\mathbf{v}_1 + \frac{11}{36}\mathbf{v}_2 + \frac{9}{18}\mathbf{v}_3 \\ &= \frac{2}{9}\mathbf{v}_1 + \frac{11}{36}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 \\ &= \left(0, \frac{2}{9}, -\frac{8}{9}, -\frac{2}{9}\right) + \left(\frac{33}{36}, \frac{55}{36}, \frac{11}{36}, \frac{11}{36}\right) + \left(\frac{1}{2}, 0, \frac{1}{2}, -2\right) \\ &= \left(\frac{17}{12}, \frac{7}{4}, -\frac{1}{12}, -\frac{23}{12}\right)\end{aligned}$$

17. (a) $\langle \mathbf{x}, \mathbf{v}_1 \rangle = 0 + \frac{2}{\sqrt{18}} + 0 + \frac{1}{\sqrt{18}} = \frac{3}{\sqrt{18}}$

$\langle \mathbf{x}, \mathbf{v}_2 \rangle = \frac{1}{2} + \frac{10}{6} + 0 - \frac{1}{6} = 2$

$\langle \mathbf{x}, \mathbf{v}_3 \rangle = \frac{1}{\sqrt{18}} + 0 + 0 + \frac{4}{\sqrt{18}} = \frac{5}{\sqrt{18}}$

$$\begin{aligned}\text{proj}_W \mathbf{x} &= \frac{3}{\sqrt{18}}\mathbf{v}_1 + 2\mathbf{v}_2 + \frac{5}{\sqrt{18}}\mathbf{v}_3 \\ &= \left(0, \frac{3}{18}, -\frac{12}{18}, -\frac{3}{18}\right) + \left(1, \frac{5}{3}, \frac{1}{3}, \frac{1}{3}\right) + \left(\frac{5}{18}, 0, \frac{5}{18}, -\frac{20}{18}\right) \\ &= \left(\frac{23}{18}, \frac{11}{6}, -\frac{1}{18}, -\frac{17}{18}\right)\end{aligned}$$

(b) $\langle \mathbf{x}, \mathbf{v}_1 \rangle = \frac{1}{2} + 1 + 0 - \frac{1}{2} = 1$

$\langle \mathbf{x}, \mathbf{v}_2 \rangle = \frac{1}{2} + 1 + 0 + \frac{1}{2} = 2$

$\langle \mathbf{x}, \mathbf{v}_3 \rangle = \frac{1}{2} - 1 + 0 + \frac{1}{2} = 0$

$$\begin{aligned}\operatorname{proj}_W \mathbf{x} &= 1\mathbf{v}_1 + 2\mathbf{v}_2 + 0\mathbf{v}_3 \\ &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + (1, 1, -1, -1) \\ &= \left(\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}\right)\end{aligned}$$

19. (a) From Exercise 14(a),

$\mathbf{w}_1 = \operatorname{proj}_W \mathbf{x} = \left(\frac{3}{2}, \frac{3}{2}, -1, -1\right)$, so

$\mathbf{w}_2 = \mathbf{x} - \operatorname{proj}_W \mathbf{x} = \left(-\frac{1}{2}, \frac{1}{2}, 1, -1\right)$.

(b) From Exercise 15(a),

$\mathbf{w}_1 = \operatorname{proj}_W \mathbf{x} = \left(\frac{7}{4}, \frac{5}{4}, -\frac{3}{4}, -\frac{5}{4}\right)$, so

$\mathbf{w}_2 = \mathbf{x} - \operatorname{proj}_W \mathbf{x} = \left(-\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, -\frac{3}{4}\right)$.

21. (a) First, transform the given basis into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

$\mathbf{v}_1 = \mathbf{u}_1 = (1, -3)$

$\|\mathbf{v}_1\| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$

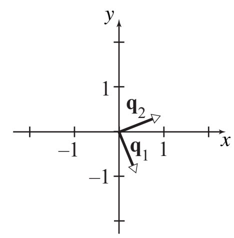
$$\begin{aligned}\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\ &= (2, 2) - \frac{2-6}{10}(1, -3) \\ &= (2, 2) - \left(-\frac{2}{5}, \frac{6}{5}\right) \\ &= \left(\frac{12}{5}, \frac{4}{5}\right)\end{aligned}$$

$$\begin{aligned}\|\mathbf{v}_2\| &= \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{4}{5}\right)^2} \\ &= \sqrt{\frac{160}{25}} \\ &= \frac{4\sqrt{10}}{5}\end{aligned}$$

The orthonormal basis is

$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, -3)}{\sqrt{10}} = \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$,

$$\begin{aligned}\mathbf{q}_2 &= \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \\ &= \frac{\left(\frac{12}{5}, \frac{4}{5}\right)}{\frac{4\sqrt{10}}{5}} \\ &= \frac{5}{4\sqrt{10}}\left(\frac{12}{5}, \frac{4}{5}\right) \\ &= \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\end{aligned}$$



(b) First, transform the given basis into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

$\mathbf{v}_1 = \mathbf{u}_1 = (1, 0)$

$\|\mathbf{v}_1\| = \sqrt{1^2 + 0^2} = 1$

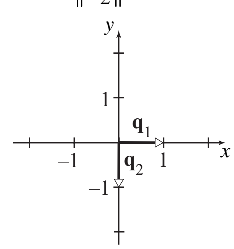
$$\begin{aligned}\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\ &= (3, -5) - \frac{3+0}{1^2}(1, 0) \\ &= (3, -5) - (3, 0) \\ &= (0, -5)\end{aligned}$$

$\|\mathbf{v}_2\| = \sqrt{0^2 + (-5)^2} = 5$

The orthonormal basis is

$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, 0)}{1} = (1, 0)$

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{(0, -5)}{5} = (0, -1)$.



23. First, transform the given basis into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

$$\mathbf{v}_1 = \mathbf{u}_1 = (0, 2, 1, 0)$$

$$\|\mathbf{v}_1\| = \sqrt{0^2+2^2+1^2+0^2} = \sqrt{5}$$

$$\begin{aligned}
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\
&= (1, -1, 0, 0) - \frac{0-2+0+0}{5}(0, 2, 1, 0) \\
&= (1, -1, 0, 0) + \left(0, \frac{4}{5}, \frac{2}{5}, 0\right) \\
&= \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)
\end{aligned}$$

$$\|\mathbf{v}_2\| = \sqrt{1^2+\left(-\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2+0^2} = \sqrt{\frac{30}{25}} = \frac{\sqrt{30}}{5}$$

$$\begin{aligned}
\mathbf{v}_3 &= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\
&= (1, 2, 0, -1) - \frac{0+4+0+0}{5}(0, 2, 1, 0) - \frac{1-\frac{2}{5}+0+0}{\frac{6}{5}}\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) \\
&= (1, 2, 0, -1) - \left(0, \frac{8}{5}, \frac{4}{5}, 0\right) - \left(\frac{1}{2}, -\frac{1}{10}, \frac{1}{5}, 0\right) \\
&= \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)
\end{aligned}$$

$$\begin{aligned}
\|\mathbf{v}_3\| &= \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+(-1)^2+(-1)^2} \\
&= \sqrt{\frac{5}{2}} \\
&= \frac{\sqrt{10}}{2}
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_4 &= \mathbf{u}_4 - \frac{\langle \mathbf{u}_4, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_4, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \frac{\langle \mathbf{u}_4, \mathbf{v}_3\rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3 \\
&= (1, 0, 0, 1) - \frac{0+0+0+0}{5}\mathbf{v}_1 - \frac{1+0+0+0}{\frac{6}{5}}\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right) - \frac{\frac{1}{2}+0+0-1}{\frac{5}{2}}\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \\
&= (1, 0, 0, 1) - \left(\frac{5}{6}, -\frac{1}{6}, \frac{1}{3}, 0\right) - \left(-\frac{1}{10}, -\frac{1}{10}, \frac{1}{5}, \frac{1}{5}\right) \\
&= \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)
\end{aligned}$$

$$\begin{aligned}
\|\mathbf{v}_4\| &= \sqrt{\left(\frac{4}{15}\right)^2+\left(\frac{4}{15}\right)^2+\left(-\frac{8}{15}\right)^2+\left(\frac{4}{5}\right)^2} \\
&= \sqrt{\frac{240}{225}} \\
&= \frac{4}{\sqrt{15}}
\end{aligned}$$

The orthonormal basis is $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(0, 2, 1, 0)}{\sqrt{5}} = \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right),$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)}{\frac{\sqrt{30}}{5}} = \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right),$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)}{\frac{\sqrt{10}}{2}} = \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right),$$

$$\mathbf{q}_4 = \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \frac{\left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right)}{\frac{4}{\sqrt{15}}} = \left(\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{2}{\sqrt{15}}, \frac{3}{\sqrt{15}}\right).$$

25. First, transform the given basis into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$$\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 1)$$

$$\begin{aligned}\|\mathbf{v}_1\| &= \sqrt{\langle \mathbf{v}_1, \mathbf{v}_1\rangle} \\ &= \sqrt{1^2 + 2(1)^2 + 3(1)^2} \\ &= \sqrt{1+2+3} \\ &= \sqrt{6}\end{aligned}$$

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\ &= (1, 1, 0) - \frac{1(1) + 2(1)(1) + 3(0)(1)}{6}(1, 1, 1) \\ &= (1, 1, 0) - \frac{1}{2}(1, 1, 1) \\ &= \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)\end{aligned}$$

$$\begin{aligned}\|\mathbf{v}_2\| &= \sqrt{\langle \mathbf{v}_2, \mathbf{v}_2\rangle} \\ &= \sqrt{\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 + 3\left(-\frac{1}{2}\right)^2} \\ &= \sqrt{6\left(\frac{1}{4}\right)} \\ &= \frac{\sqrt{6}}{2}\end{aligned}$$

$$\begin{aligned}\mathbf{v}_3 &= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\ &= (1, 0, 0) - \frac{1+0+0}{6}(1, 1, 1) - \frac{\frac{1}{2}+0+0}{\frac{6}{4}}\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \\ &= (1, 0, 0) - \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) - \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}\right) \\ &= \left(\frac{2}{3}, -\frac{1}{3}, 0\right)\end{aligned}$$

$$\|\mathbf{v}_3\| = \sqrt{\langle \mathbf{v}_3, \mathbf{v}_3\rangle} = \sqrt{\left(\frac{2}{3}\right)^2 + 2\left(-\frac{1}{3}\right)^2 + 3(0)^2} = \sqrt{\frac{4}{9}+\frac{2}{9}} = \frac{\sqrt{6}}{3}$$

The orthonormal basis is

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, 1, 1)}{\sqrt{6}} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right),$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)}{\frac{\sqrt{6}}{2}} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right),$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left(\frac{2}{3}, -\frac{1}{3}, 0\right)}{\frac{\sqrt{6}}{3}} = \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0\right).$$

27. Let $W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

$$\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 1)$$

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\ &= (2, 0, -1) - \frac{2+0-1}{1^2+1^2+1^2}(1, 1, 1) \\ &= (2, 0, -1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ &= \left(\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}\right)\end{aligned}$$

$\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for W.

$$\|\mathbf{v}_1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\|\mathbf{v}_2\| = \sqrt{\left(\frac{5}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{4}{3}\right)^2} = \sqrt{\frac{42}{9}} = \frac{\sqrt{42}}{3}$$

$$\begin{aligned}\mathbf{w}_1 &= \text{proj}_W \mathbf{w} \\ &= \frac{\langle \mathbf{w}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{w}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\ &= \frac{1+2+3}{3}\mathbf{v}_1 + \frac{\frac{5}{3}-\frac{2}{3}-\frac{12}{3}}{\frac{42}{9}}\mathbf{v}_2 \\ &= 2(1, 1, 1) - \frac{9}{14}\left(\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}\right) \\ &= (2, 2, 2) - \left(\frac{15}{14}, -\frac{3}{14}, -\frac{6}{7}\right) \\ &= \left(\frac{13}{14}, \frac{31}{14}, \frac{20}{7}\right)\end{aligned}$$

$$\begin{aligned}\mathbf{w}_2 &= \mathbf{w} - \mathbf{w}_1 \\ &= (1, 2, 3) - \left(\frac{13}{14}, \frac{31}{14}, \frac{20}{7}\right) \\ &= \left(\frac{1}{14}, -\frac{3}{14}, \frac{1}{7}\right)\end{aligned}$$

29. (a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Since $\det(A) = 3 + 2 = 5 \neq 0$, A is invertible, so it has a QR-decomposition.

Let $\mathbf{u}_1 = (1, 2)$ and $\mathbf{u}_2 = (-1, 3)$.

Apply the Gram-Schmidt process with normalization to $\mathbf{u}_1$ and $\mathbf{u}_2$.

$$\mathbf{v}_1 = \mathbf{u}_1 = (1, 2)$$

$$\|\mathbf{v}_1\| = \sqrt{1^2+2^2} = \sqrt{5}$$

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\ &= (-1, 3) - \frac{-1+6}{5}(1, 2) \\ &= (-1, 3) - (1, 2) \\ &= (-2, 1)\end{aligned}$$

$$\|\mathbf{v}_2\| = \sqrt{(-2)^2+1^2} = \sqrt{5}$$

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, 2)}{\sqrt{5}} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{(-2, 1)}{\sqrt{5}} = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

$$\langle \mathbf{u}_1, \mathbf{q}_1\rangle = \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} = \sqrt{5}$$

$$\langle \mathbf{u}_2, \mathbf{q}_1\rangle = -\frac{1}{\sqrt{5}} + \frac{6}{\sqrt{5}} = \sqrt{5}$$

$$\langle \mathbf{u}_2, \mathbf{q}_2\rangle = \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}} = \sqrt{5}$$

The QR-decomposition of A is

$$\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix} = [\mathbf{q}_1 \quad \mathbf{q}_2]\begin{bmatrix}\langle \mathbf{u}_1, \mathbf{q}_1\rangle & \langle \mathbf{u}_2, \mathbf{q}_1\rangle\\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2\rangle\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix}\begin{bmatrix}\sqrt{5} & \sqrt{5}\\ 0 & \sqrt{5}\end{bmatrix}.$$

(b) $A = \begin{bmatrix}1 & 2\\ 0 & 1\\ 1 & 4\end{bmatrix}$

By inspection, the column vectors of A are linearly independent, so A has a QR-decomposition. Let $\mathbf{u}_1 = (1, 0, 1)$ and $\mathbf{u}_2 = (2, 1, 4)$.

Apply the Gram-Schmidt process with normalization to $\mathbf{u}_1$ and $\mathbf{u}_2$.

$\mathbf{v}_1 = \mathbf{u}_1 = (1, 0, 1)$

$\|\mathbf{v}_1\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1\\ &= (2, 1, 4) - \frac{2+0+4}{2}(1, 0, 1)\\ &= (2, 1, 4) - (3, 0, 3)\\ &= (-1, 1, 1)\end{aligned}$$

$\|\mathbf{v}_2\| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$

$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, 0, 1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{(-1, 1, 1)}{\sqrt{3}} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\langle \mathbf{u}_1, \mathbf{q}_1\rangle = \frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} = \sqrt{2}$

$\langle \mathbf{u}_2, \mathbf{q}_1\rangle = \frac{2}{\sqrt{2}} + 0 + \frac{4}{\sqrt{2}} = 3\sqrt{2}$

$\langle \mathbf{u}_2, \mathbf{q}_2\rangle = -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}} = \sqrt{3}$

The QR-decomposition of A is

$$\begin{bmatrix}1 & 2\\ 0 & 1\\ 1 & 4\end{bmatrix} = [\mathbf{q}_1 \quad \mathbf{q}_2]\begin{bmatrix}\langle \mathbf{u}_1, \mathbf{q}_1\rangle & \langle \mathbf{u}_2, \mathbf{q}_1\rangle\\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2\rangle\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}}\\ 0 & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 3\sqrt{2}\\ 0 & \sqrt{3}\end{bmatrix}$$

(c) $A = \begin{bmatrix}1 & 1\\ -2 & 1\\ 2 & 1\end{bmatrix}$

By inspection, the column vectors of A are linearly independent, so A has a QR-decomposition.

Let $\mathbf{u}_1 = (1, -2, 2)$ and $\mathbf{u}_2 = (1, 1, 1)$.

Apply the Gram-Schmidt process with normalization to $\mathbf{u}_1$ and $\mathbf{u}_2$.

$\mathbf{v}_1 = \mathbf{u}_1 = (1, -2, 2)$

$\|\mathbf{v}_1\| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1\\ &= (1, 1, 1) - \frac{1-2+2}{9}(1, -2, 2)\\ &= (1, 1, 1) - \left(\frac{1}{9}, -\frac{2}{9}, \frac{2}{9}\right)\\ &= \left(\frac{8}{9}, \frac{11}{9}, \frac{7}{9}\right)\end{aligned}$$

$$\begin{aligned}\|\mathbf{v}_2\| &= \sqrt{\left(\frac{8}{9}\right)^2 + \left(\frac{11}{9}\right)^2 + \left(\frac{7}{9}\right)^2}\\ &= \sqrt{\frac{234}{81}}\\ &= \frac{\sqrt{234}}{9}\end{aligned}$$

$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, -2, 2)}{3} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$

$$\begin{aligned}\mathbf{q}_2 &= \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}\\ &= \frac{\left(\frac{8}{9}, \frac{11}{9}, \frac{7}{9}\right)}{\frac{\sqrt{234}}{9}}\\ &= \left(\frac{8}{\sqrt{234}}, \frac{11}{\sqrt{234}}, \frac{7}{\sqrt{234}}\right)\end{aligned}$$

$\langle \mathbf{u}_1, \mathbf{q}_1\rangle = \frac{1}{3} + \frac{4}{3} + \frac{4}{3} = 3$

$\langle \mathbf{u}_2, \mathbf{q}_1\rangle = \frac{1}{3} - \frac{2}{3} + \frac{2}{3} = \frac{1}{3}$

$$\begin{aligned}\langle \mathbf{u}_2, \mathbf{q}_2\rangle &= \frac{8}{\sqrt{234}} + \frac{11}{\sqrt{234}} + \frac{7}{\sqrt{234}}\\ &= \frac{26}{3\sqrt{26}}\\ &= \frac{\sqrt{26}}{3}\end{aligned}$$

The QR-decomposition of A is

$$\begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 1 & 1 \end{bmatrix} = [\mathbf{q}_1 \quad \mathbf{q}_2]\begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{8}{\sqrt{234}} \\ -\frac{2}{3} & \frac{11}{\sqrt{234}} \\ \frac{2}{3} & \frac{7}{\sqrt{234}} \end{bmatrix}\begin{bmatrix} 3 & \frac{1}{3} \\ 0 & \frac{\sqrt{26}}{3} \end{bmatrix}$$

(d) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

Since $\det(A) = -4 \neq 0$, A is invertible, so it has a QR-decomposition. Let $\mathbf{u}_1 = (1, 0, 1)$, $\mathbf{u}_2 = (0, 1, 2)$, and $\mathbf{u}_3 = (2, 1, 0)$.

Apply the Gram-Schmidt process with normalization to $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

$\mathbf{v}_1 = \mathbf{u}_1 = (1, 0, 1)$

$\|\mathbf{v}_1\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (0, 1, 2) - \frac{0+0+2}{2}(1, 0, 1) = (0, 1, 2) - (1, 0, 1) = (-1, 1, 1)$$

$\|\mathbf{v}_2\| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\ &= (2, 1, 0) - \frac{2+0+0}{2}(1, 0, 1) - \frac{-2+1+0}{3}(-1, 1, 1) \\ &= (2, 1, 0) - (1, 0, 1) + \left(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ &= \left(\frac{2}{3}, \frac{4}{3}, -\frac{2}{3}\right) \end{aligned}$$

$$\|\mathbf{v}_3\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{24}{9}} = \frac{2\sqrt{6}}{3}$$

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, 0, 1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{(-1, 1, 1)}{\sqrt{3}} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left(\frac{2}{3}, \frac{4}{3}, -\frac{2}{3}\right)}{\frac{2\sqrt{6}}{3}} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$$

$$\langle \mathbf{u}_1, \mathbf{q}_1 \rangle = \frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$\langle \mathbf{u}_2, \mathbf{q}_1 \rangle = 0 + 0 + \frac{2}{\sqrt{2}} = \sqrt{2}$$

$\langle \mathbf{u}_3, \mathbf{q}_1 \rangle = \frac{2}{\sqrt{2}} + 0 + 0 = \sqrt{2}$

$\langle \mathbf{u}_2, \mathbf{q}_2 \rangle = 0 + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \sqrt{3}$

$\langle \mathbf{u}_3, \mathbf{q}_2 \rangle = -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0 = -\frac{1}{\sqrt{3}}$

$\langle \mathbf{u}_3, \mathbf{q}_3 \rangle = \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} + 0 = \frac{4}{\sqrt{6}}$

The QR-decomposition of A is

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3] \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \langle \mathbf{u}_3, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \langle \mathbf{u}_3, \mathbf{q}_2 \rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{4}{\sqrt{6}} \end{bmatrix}.$$

(e) $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

Since $\det(A) = -1 \neq 0$, A is invertible, so it has a QR-decomposition.
Let $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (2, 1, 3)$, and $\mathbf{u}_3 = (1, 1, 1)$.

Apply the Gram-Schmidt process with normalization to $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

$\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 0)$

$\|\mathbf{v}_1\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$

$$\begin{aligned} \mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 \\ &= (2, 1, 3) - \frac{2+1+0}{2}(1, 1, 0) \\ &= (2, 1, 3) - \left(\frac{3}{2}, \frac{3}{2}, 0\right) \\ &= \left(\frac{1}{2}, -\frac{1}{2}, 3\right) \end{aligned}$$

$\|\mathbf{v}_2\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 3^2} = \sqrt{\frac{19}{2}}$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$
$$= (1, 1, 1) - \frac{1+1+0}{2}(1, 1, 0) - \frac{\frac{1}{2}-\frac{1}{2}+3}{\frac{19}{2}}\left(\frac{1}{2}, -\frac{1}{2}, 3\right)$$
$$= (1, 1, 1) - (1, 1, 0) - \left(\frac{3}{19}, -\frac{3}{19}, \frac{18}{19}\right)$$
$$= \left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19}\right)$$

$$\|\mathbf{v}_3\| = \sqrt{\left(-\frac{3}{19}\right)^2 + \left(\frac{3}{19}\right)^2 + \left(\frac{1}{19}\right)^2} = \sqrt{\frac{1}{19}} = \frac{1}{\sqrt{19}}$$

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, 1, 0)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{\left(\frac{1}{2}, -\frac{1}{2}, 3\right)}{\sqrt{\frac{19}{2}}} = \left(\frac{\sqrt{2}}{2\sqrt{19}}, -\frac{\sqrt{2}}{2\sqrt{19}}, \frac{3\sqrt{2}}{\sqrt{19}}\right)$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left(-\frac{3}{19}, \frac{3}{19}, \frac{1}{19}\right)}{\frac{1}{\sqrt{19}}} = \left(-\frac{3}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}}\right)$$

$$\langle \mathbf{u}_1, \mathbf{q}_1\rangle = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 = \sqrt{2}$$

$$\langle \mathbf{u}_2, \mathbf{q}_1\rangle = \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 = \frac{3}{\sqrt{2}}$$

$$\langle \mathbf{u}_3, \mathbf{q}_1\rangle = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 = \sqrt{2}$$

$$\langle \mathbf{u}_2, \mathbf{q}_2\rangle = \frac{2\sqrt{2}}{2\sqrt{19}} - \frac{\sqrt{2}}{2\sqrt{19}} + \frac{9\sqrt{2}}{\sqrt{19}} = \frac{\sqrt{19}}{\sqrt{2}}$$

$$\langle \mathbf{u}_3, \mathbf{q}_2\rangle = \frac{\sqrt{2}}{2\sqrt{19}} - \frac{\sqrt{2}}{2\sqrt{19}} + \frac{3\sqrt{2}}{\sqrt{19}} = \frac{3\sqrt{2}}{\sqrt{19}}$$

$$\langle \mathbf{u}_3, \mathbf{q}_3\rangle = -\frac{3}{\sqrt{19}} + \frac{3}{\sqrt{19}} + \frac{1}{\sqrt{19}} = \frac{1}{\sqrt{19}}$$

The QR-decomposition of A is

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3]\begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1\rangle & \langle \mathbf{u}_2, \mathbf{q}_1\rangle & \langle \mathbf{u}_3, \mathbf{q}_1\rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2\rangle & \langle \mathbf{u}_3, \mathbf{q}_2\rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3\rangle \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}\begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$$

(f) $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3]$

By inspection, $\mathbf{c}_3 - \mathbf{c}_1 = 2\mathbf{c}_2$, so the column vectors of A are not linearly independent and A does not have a QR-decomposition.

31. The diagonal entries of R are $\langle \mathbf{u}_i, \mathbf{q}_i \rangle$ for $i = 1, 2, ..., n$, where $\mathbf{q}_i = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$ is the normalization of a vector $\mathbf{v}_i$ that is the result of applying the Gram-Schmidt process to $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$. Thus, $\mathbf{v}_i$ is $\mathbf{u}_i$ minus a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{i-1}$, so $\mathbf{u}_i = \mathbf{v}_i + k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_{i-1}\mathbf{v}_{i-1}$. Thus, $\langle \mathbf{u}_i, \mathbf{v}_i \rangle = \langle \mathbf{v}_i, \mathbf{v}_i \rangle$ and

$$\langle \mathbf{u}_i, \mathbf{q}_i \rangle = \left\langle \mathbf{u}_i, \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|} \right\rangle = \frac{1}{\|\mathbf{v}_i\|}\langle \mathbf{v}_i, \mathbf{v}_i \rangle = \|\mathbf{v}_i\|.$$

Since each vector $\mathbf{v}_i$ is nonzero, each diagonal entry of R is nonzero.

33. First transform the basis $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \{1, x, x^2\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$\mathbf{v}_1 = \mathbf{p}_1 = 1$

$\|\mathbf{v}_1\|^2 = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \int_0^1 1^2\,dx = x\big|_0^1 = 1$

$\|\mathbf{v}_1\| = \sqrt{1} = 1$

$\langle \mathbf{p}_2, \mathbf{v}_1 \rangle = \int_0^1 x \cdot 1\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$

$\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = x - \frac{\frac{1}{2}}{1}(1) = -\frac{1}{2} + x$

$$\begin{aligned} \|\mathbf{v}_2\|^2 &= \langle \mathbf{v}_2, \mathbf{v}_2 \rangle \\ &= \int_0^1 \left(-\frac{1}{2} + x\right)^2 dx \\ &= \int_0^1 \left(\frac{1}{4} - x + x^2\right) dx \\ &= \left(\frac{1}{4}x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)\Big|_0^1 \\ &= \frac{1}{12} \end{aligned}$$

$\|\mathbf{v}_2\| = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}}$

$\langle \mathbf{p}_3, \mathbf{v}_1 \rangle = \int_0^1 x^2 \cdot 1\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$

$$\begin{aligned} \langle \mathbf{p}_3, \mathbf{v}_2 \rangle &= \int_0^1 x^2\left(-\frac{1}{2} + x\right) dx \\ &= \int_0^1 \left(-\frac{1}{2}x^2 + x^3\right) dx \\ &= \left(-\frac{1}{6}x^3 + \frac{1}{4}x^3\right)\Big|_0^1 \\ &= \frac{1}{12} \end{aligned}$$

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{p}_3 - \frac{\langle \mathbf{p}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{p}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\ &= x^2 - \frac{\frac{1}{3}}{1}(1) - \frac{\frac{1}{12}}{\frac{1}{12}}\left(-\frac{1}{2} + x\right) \\ &= x^2 - \frac{1}{3} + \frac{1}{2} - x \\ &= \frac{1}{6} - x + x^2 \end{aligned}$$

$$\begin{aligned} \|\mathbf{v}_3\|^2 &= \langle \mathbf{v}_3, \mathbf{v}_3 \rangle \\ &= \int_0^1 \left(\frac{1}{6} - x + x^2\right)^2 dx \\ &= \int_0^1 \left(\frac{1}{36} - \frac{1}{3}x + \frac{4}{3}x^2 - 2x^3 + x^4\right) dx \\ &= \left(\frac{1}{36}x - \frac{1}{6}x^2 + \frac{4}{9}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right)\Big|_0^1 \\ &= \frac{1}{180} \end{aligned}$$

$\|\mathbf{v}_3\| = \sqrt{\frac{1}{180}} = \frac{1}{6\sqrt{5}}$

The orthonormal basis is

$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{1} = 1,$

$$\begin{aligned} \mathbf{q}_2 &= \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \\ &= \frac{-\frac{1}{2} + x}{\frac{1}{2\sqrt{3}}} \\ &= 2\sqrt{3}\left(-\frac{1}{2} + x\right) \\ &= \sqrt{3}(-1 + 2x) \end{aligned}$$

$$\begin{aligned}\mathbf{q}_3 &= \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}\\ &= \frac{\frac{1}{6}-x+x^2}{\frac{1}{6\sqrt{5}}}\\ &= 6\sqrt{5}\left(\frac{1}{6}-x+x^2\right)\\ &= \sqrt{5}(1-6x+6x^2)\end{aligned}$$

True/False 6.3

(a) False; for example, the vectors (1, 2) and (−1, 3) in R^2 are linearly independent but not orthogonal.

(b) False; the vectors must be nonzero for this to be true.

(c) True; a nontrivial subspace of R^3 will have a basis, which can be transformed into an orthonormal basis with respect to the Euclidean inner product.

(d) True; a nonzero finite-dimensional inner product space will have finite basis which can be transformed into an orthonormal basis with respect to the inner product via the Gram-Schmidt process with normalization.

(e) False; $\text{proj}_W \mathbf{x}$ is a vector in *W*.

(f) True; every invertible $n \times n$ matrix has a *QR*-decomposition.

Section 6.4

Exercise Set 6.4

1. (a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

The associated normal system is

$$\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix}$

$$\begin{aligned}A^TA &= \begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 4 & 2 \\ 0 & 2 & 5 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix}\\ &= \begin{bmatrix} 15 & -1 & 5 \\ -1 & 22 & 30 \\ 5 & 30 & 45 \end{bmatrix}\end{aligned}$$

$$A^T\mathbf{b} = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 4 & 2 \\ 0 & 2 & 5 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 13 \end{bmatrix}$$

The associated normal system is

$$\begin{bmatrix} 15 & -1 & 5 \\ -1 & 22 & 30 \\ 5 & 30 & 45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 13 \end{bmatrix}$$

3. (a) $A^TA = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

The associated normal system is

$$\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}.$$

$\begin{bmatrix} 3 & -2 & 14 \\ -2 & 6 & -7 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$, so the least squares solution is $x_1 = 5$, $x_2 = \frac{1}{2}$.

(b) $A^TA = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ -9 \end{bmatrix}$$

The associated normal system is

$$\begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ -9 \end{bmatrix}.$$

$\begin{bmatrix} 7 & 4 & -6 & 18 \\ 4 & 3 & -3 & 12 \\ -6 & -3 & 6 & -9 \end{bmatrix}$ reduces to

$\begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{bmatrix}$ so the least squares solution is $x_1 = 12,\ x_2 = -3,\ x_3 = 9.$

5. **(a)** $\mathbf{e} = \mathbf{b} - A\mathbf{x}$

$$= \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} - \begin{bmatrix} \frac{11}{2} \\ -\frac{9}{2} \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} \\ \frac{9}{2} \\ -3 \end{bmatrix}$$

$A^T\mathbf{e} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{3}{2} \\ \frac{9}{2} \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $\mathbf{e}$ is orthogonal to the column space of A.

(b) $\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 12 \\ -3 \\ 9 \end{bmatrix}$

$$= \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 9 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -3 \\ 0 \\ 3 \end{bmatrix}$$

$A^T\mathbf{e} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $\mathbf{e}$ is orthogonal to the column space of A.

7. **(a)** $A^TA = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 8 \\ 8 & 6 \end{bmatrix}$

$A^T\mathbf{b} = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$

The associated normal system is

$$\begin{bmatrix} 24 & 8 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}.$$

$\begin{bmatrix} 24 & 8 & 12 \\ 8 & 6 & 8 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & \frac{1}{10} \\ 0 & 1 & \frac{6}{5} \end{bmatrix}$, so the least squares solution is $x_1 = \dfrac{1}{10}$, $x_2 = \dfrac{6}{5}$ or $\mathbf{x} = \begin{bmatrix} \frac{1}{10} \\ \frac{6}{5} \end{bmatrix}$.

The error vector is

$\mathbf{e} = \mathbf{b} - A\mathbf{x}$

$$= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{10} \\ \frac{6}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{7}{5} \\ \frac{14}{5} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{5} \\ -\frac{4}{5} \\ 0 \end{bmatrix}$$

so the least squares error is

$$\|\mathbf{e}\| = \sqrt{\left(\frac{8}{5}\right)^2 + \left(-\frac{4}{5}\right)^2 + 0^2} = \sqrt{\frac{80}{25}} = \frac{4}{5}\sqrt{5}.$$

(b) $A^TA = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & -6 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 14 & 42 \\ 42 & 126 \end{bmatrix}$

$A^T\mathbf{b} = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$

The associated normal system is

$$\begin{bmatrix} 14 & 42 \\ 42 & 126 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}.$$

$\begin{bmatrix} 14 & 42 & 4 \\ 42 & 126 & 12 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 3 & \frac{2}{7} \\ 0 & 0 & 0 \end{bmatrix}$, so the least squares solutions are $x_1 = \frac{2}{7} - 3t$,

$x_2 = t$ or $\mathbf{x} = \begin{bmatrix} \frac{2}{7} \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ for t a real number. The error vector is

$$\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -2 & -6 \\ 3 & 9 \end{bmatrix}\left(\begin{bmatrix} \frac{2}{7} \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ \frac{6}{7} \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{5}{7} \\ \frac{4}{7} \\ \frac{1}{7} \end{bmatrix}.$$

The least squares error is

$$\|\mathbf{e}\| = \sqrt{\left(\frac{5}{7}\right)^2 + \left(\frac{4}{7}\right)^2 + \left(\frac{1}{7}\right)^2} = \sqrt{\frac{42}{49}} = \frac{1}{7}\sqrt{42}.$$

(c) $$A^TA = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4 \\ -1 & 11 & 10 \\ 4 & 10 & 14 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} -7 \\ 14 \\ 7 \end{bmatrix}$$

The associated normal system is

$$\begin{bmatrix} 5 & -1 & 4 \\ -1 & 11 & 10 \\ 4 & 10 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 14 \\ 7 \end{bmatrix}.$$

$\begin{bmatrix} 5 & -1 & 4 & -7 \\ -1 & 11 & 10 & 14 \\ 4 & 10 & 14 & 7 \end{bmatrix}$ reduces to

$\begin{bmatrix} 1 & 0 & 1 & -\frac{7}{6} \\ 0 & 1 & 1 & \frac{7}{6} \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the least squares

solutions are $x_1 = -\frac{7}{6} - t$, $x_2 = \frac{7}{6} - t$,

$x_3 = t$ or $\mathbf{x} = \begin{bmatrix} -\frac{7}{6} \\ \frac{7}{6} \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ for t a real number.

The error vector is

$$\mathbf{e} = \mathbf{b} - A\mathbf{x}$$

$$= \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} - \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}\left(\begin{bmatrix} -\frac{7}{6} \\ \frac{7}{6} \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} - \left(\begin{bmatrix} \frac{14}{3} \\ -\frac{7}{6} \\ \frac{7}{6} \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{7}{3} \\ \frac{7}{6} \\ -\frac{49}{6} \end{bmatrix}.$$

The least squares error is

$$\|\mathbf{e}\| = \sqrt{\left(\frac{7}{3}\right)^2 + \left(\frac{7}{6}\right)^2 + \left(-\frac{49}{6}\right)^2} = \sqrt{\frac{294}{4}} = \frac{7}{2}\sqrt{6}.$$

9. (a) Let $A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$.

$$A^TA = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}$$

$\det(A^TA) = 3 \neq 0$, so the column vectors of A are linearly independent and the column space of A, W, is the subspace of R^4 spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$$(A^TA)^{-1} = \begin{bmatrix} 3 & -2 & 2 \\ -2 & 2 & -1 \\ 2 & -1 & \frac{5}{3} \end{bmatrix}$$

$$\text{proj}_W \mathbf{u} = A(A^TA)^{-1}A^T\mathbf{u}$$

$$= \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 2 \\ -2 & 2 & -1 \\ 2 & -1 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 9 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 2 \\ 9 \\ 5 \end{bmatrix}$$

$\text{proj}_W \mathbf{u} = (7, 2, 9, 5)$

(b) Let $A = \begin{bmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

$$A^TA = \begin{bmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{bmatrix}$$

$\det(A^TA) = 10 \neq 0$ so the column vectors of A are linearly independent and the column space of A, W, is the subspace of R^4 spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$$(A^TA)^{-1} = \begin{bmatrix} \frac{68}{5} & \frac{86}{5} & -\frac{31}{5} \\ \frac{86}{5} & \frac{219}{10} & -\frac{79}{10} \\ -\frac{31}{5} & -\frac{79}{10} & \frac{29}{10} \end{bmatrix}$$

$$\text{proj}_W \mathbf{u} = A(A^TA)^{-1}A^T\mathbf{u}$$

$$= \begin{bmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{68}{5} & \frac{86}{5} & -\frac{31}{5} \\ \frac{86}{5} & \frac{219}{10} & -\frac{79}{10} \\ -\frac{31}{5} & -\frac{79}{10} & \frac{29}{10} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{12}{5} \\ -\frac{4}{5} \\ \frac{12}{5} \\ \frac{16}{5} \end{bmatrix}$$

$$\text{proj}_W \mathbf{u} = \left(-\frac{12}{5}, -\frac{4}{5}, \frac{12}{5}, \frac{16}{5}\right)$$

11. (a) $A^TA = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4 \\ -1 & 11 & 10 \\ 4 & 10 & 14 \end{bmatrix}$

$\det(A^TA) = 0,$ so A does not have linearly independent column vectors.

(b) $A^TA = \begin{bmatrix} 2 & 0 & -1 & 4 \\ -1 & 1 & 0 & -5 \\ 3 & 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ -1 & 0 & -2 \\ 4 & -5 & 3 \end{bmatrix} = \begin{bmatrix} 21 & -22 & 20 \\ -22 & 27 & -17 \\ 20 & -17 & 23 \end{bmatrix}$

$\det(A^TA) = 0,$ so A does not have linearly independent column vectors.

13. (a) W = the xz-plane is two-dimensional and one basis for W is $\{\mathbf{w}_1, \mathbf{w}_2\}$ where $\mathbf{w}_1 = (1, 0, 0)$, $\mathbf{w}_2 = (0, 0, 1)$.

Thus $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned}[P] &= A(A^TA)^{-1}A^T \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

(b) W = the yz-plane is two-dimensional and one basis for W is $\{\mathbf{w}_1, \mathbf{w}_2\}$ where $\mathbf{w}_1 = (0, 1, 0)$ and $\mathbf{w}_2 = (0, 0, 1)$.

Thus $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$A^TA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned}[P] &= A(A^TA)^{-1}A^T \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

15. (a) If $x = s$ and $y = t$, then a point on the plane is $(s, t, -5s + 3t) = s(1, 0, -5) + t(0, 1, 3)$. $\mathbf{w}_1 = (1, 0, -5)$ and $\mathbf{w}_2 = (0, 1, 3)$ are a basis for W.

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix}$,

$A^TA = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 26 & -15 \\ -15 & 10 \end{bmatrix}$

$\det(A^TA) = 35,\ (A^TA)^{-1} = \frac{1}{35}\begin{bmatrix} 10 & 15 \\ 15 & 26 \end{bmatrix}$

$$\begin{aligned}[P] &= A(A^TA)^{-1}A^T \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix}\left(\frac{1}{35}\begin{bmatrix} 10 & 15 \\ 15 & 26 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix} \\ &= \frac{1}{35}\begin{bmatrix} 10 & 15 \\ 15 & 26 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix} \\ &= \frac{1}{35}\begin{bmatrix} 10 & 15 & -5 \\ 15 & 26 & 3 \\ -5 & 3 & 34 \end{bmatrix}\end{aligned}$$

(c) $[P]\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 10 & 15 & -5 \\ 15 & 26 & 3 \\ -5 & 3 & 34 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

$= \frac{1}{35}\begin{bmatrix} 10x_0 + 15y_0 - 5z_0 \\ 15x_0 + 26y_0 + 3z_0 \\ -5x_0 + 3y_0 + 34z_0 \end{bmatrix}$

$= \begin{bmatrix} \frac{10x_0+15y_0-5z_0}{35} \\ \frac{15x_0+26y_0+3z_0}{35} \\ \frac{-5x_0+3y_0+34z_0}{35} \end{bmatrix}$

The orthogonal projection is $\left(\frac{2x_0 - 3y_0 - z_0}{7}, \frac{15x_0 + 26y_0 + 3z_0}{35}, \frac{-5x_0 + 3y_0 + 34z_0}{35}\right)$.

(d) The orthogonal projection of $P_0(1, -2, 4)$ on W is

$\left(\frac{2(1)-3(-2)-4}{7}, \frac{15(1)+26(-2)+3(4)}{35}, \frac{-5(1)+3(-2)+34(4)}{35}\right) = \left(-\frac{8}{7}, -\frac{5}{7}, \frac{25}{7}\right)$.

The distance between $P_0(1, -2, 4)$ and W is $d = \sqrt{\left(1-\left(-\frac{8}{7}\right)\right)^2 + \left(-2-\left(-\frac{5}{7}\right)\right)^2 + \left(4-\frac{25}{7}\right)^2} = \frac{3\sqrt{35}}{7}$.

Using Theorem 3.3.4, the distance is $\frac{|5(1)-3(-2)+4|}{\sqrt{5^2+(-3)^2+1^2}} = \frac{15}{\sqrt{35}} = \frac{3\sqrt{35}}{7}$.

17. By inspection, when $t = 1$, the point $(t, t, t) = (1, 1, 1)$ is on line l. When $s = 1$, the point $(s, 2s - 1, 1) = (1, 1, 1)$ is on line m. Thus since $\|\mathbf{P}-\mathbf{Q}\| \geq 0$, these are the values of s and t that minimize the distance between the lines.

19. If A has linearly independent column vectors, then A^TA is invertible and the least squares solution of $A\mathbf{x} = \mathbf{b}$ is the solution of $A^TA\mathbf{x} = A^T\mathbf{b}$, but since $\mathbf{b}$ is orthogonal to the column space of A, $A^T\mathbf{b} = \mathbf{0}$, so $\mathbf{x}$ is a solution of $A^TA\mathbf{x} = \mathbf{0}$. Thus, $\mathbf{x} = \mathbf{0}$ since A^TA is invertible.

21. A^T will have linearly independent column vectors, and the column space of A^T is the row space of A. Thus, the standard matrix for the orthogonal projection of R^n onto the row space of A is
$[P] = A^T[(A^T)^TA^T]^{-1}(A^T)^T = A^T(AA^T)^{-1}A$.

True/False 6.4

(a) True; A^TA is an $n \times n$ matrix.

(b) False; only square matrices have inverses, but A^TA can be invertible when A is not a square matrix.

(c) True; if A is invertible, so is A^T, so the product A^TA is also invertible.

(d) True

(e) False; the system $A^TA\mathbf{x} = A^T\mathbf{b}$ may be consistent.

(f) True

(g) False; the least squares solution may involve a parameter.

(h) True; if A has linearly independent column vectors, then A^TA is invertible, so $A^TA\mathbf{x} = A^T\mathbf{b}$ has a unique solution.

Section 6.5

Exercise Set 6.5

1. $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, $M^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$,

$$M^TM = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix},$$

$$(M^TM)^{-1} = \frac{1}{15-9}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\mathbf{v}^* = (M^TM)^{-1}M^T\mathbf{y}$$

$$= \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ \frac{7}{2} \end{bmatrix}$$

The desired line is $y = -\frac{1}{2} + \frac{7}{2}x.$

3. $M = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}$

$$\mathbf{y} = \begin{bmatrix} 0 \\ -10 \\ -48 \\ -76 \end{bmatrix}$$

$$M^TM = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \\ 4 & 9 & 25 & 36 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 16 & 74 \\ 16 & 74 & 376 \\ 74 & 376 & 2018 \end{bmatrix}$$

$$(M^TM)^{-1} = \frac{1}{90}\begin{bmatrix} 1989 & -1116 & 135 \\ -1116 & 649 & -80 \\ 135 & -80 & 10 \end{bmatrix}$$

$$\mathbf{v}^* = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = (M^TM)^{-1}M^T\mathbf{y} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$$

The desired quadratic is $y = 2 + 5x - 3x^2$.

5. The two column vectors of M are linearly independent if and only neither is a nonzero multiple of the other. Since all the entries in the first column are equal, the columns are linearly independent if and only if the second column has at least two different entries, i.e., if and only if at least two of the numbers $x_1, x_2, \ldots, x_n$ are distinct.

11. With the substitution $X = \frac{1}{x}$, the problem becomes to find a line of the form $y = a + b \cdot X$ that best fits the data points (1, 7), $\left(\frac{1}{3}, 3\right)$, $\left(\frac{1}{6}, 1\right)$.

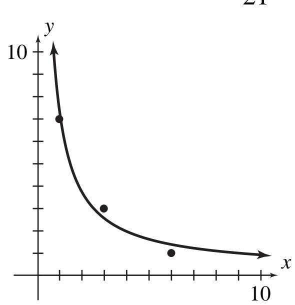
$$M = \begin{bmatrix} 1 & 1 \\ 1 & \frac{1}{3} \\ 1 & \frac{1}{6} \end{bmatrix},\ \mathbf{y} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix},$$

$$M^TM = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{3} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & \frac{1}{3} \\ 1 & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 3 & \frac{3}{2} \\ \frac{3}{2} & \frac{41}{36} \end{bmatrix},$$

$$(M^TM)^{-1} = \frac{1}{42}\begin{bmatrix} 41 & -54 \\ -54 & 108 \end{bmatrix}$$

$$\mathbf{v}^* = \begin{bmatrix} a \\ b \end{bmatrix} = (M^TM)^{-1}M^T\mathbf{y} = \begin{bmatrix} \frac{5}{21} \\ \frac{48}{7} \end{bmatrix}.$$

The line in terms of X is $y = \frac{5}{21} + \frac{48}{7}X$, so the required curve is $y = \frac{5}{21} + \frac{48}{7x}$.



True/False 6.5

(a) False; there is only a unique least squares straight line fit if the data points do not lie on a vertical line.

(b) True; if the points are not collinear, there is no solution to the system.

(c) False; the sum $d_1^2 + d_2^2 + \cdots + d_n^2$ is minimized, not the individual terms d_i.

(d) True

Section 6.6

Exercise Set 6.6

1. With $f(x) = 1 + x$:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}(1+x)dx = \frac{1}{\pi}\left(x + \frac{x^2}{2}\right)\Bigg|_0^{2\pi} = 2 + 2\pi$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_0^{2\pi}(1+x)\cos kx\,dx \\ &= \frac{1}{\pi}\int_0^{2\pi}\cos kx\,dx + \frac{1}{\pi}\int_0^{2\pi}x\cos kx\,dx \\ &= \frac{1}{k\pi}\sin kx\Big|_0^{2\pi} + 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_0^{2\pi}(1+x)\sin kx\,dx \\ &= \frac{1}{\pi}\int_0^{2\pi}\sin kx\,dx + \frac{1}{\pi}\int_0^{2\pi}x\sin kx\,dx \\ &= -\frac{1}{k\pi}\cos kx\Big|_0^{2\pi} - \frac{2}{k} \\ &= -\frac{2}{k} \end{aligned}$$

(a) $a_0 = 2 + 2\pi,\ a_1 = a_2 = 0,\ b_1 = -2,\ b_2 = -1$

Thus $f(x) \approx \frac{1}{2}(2+2\pi) - 2\sin x - \sin 2x$
$= (1+\pi) - 2\sin x - \sin 2x.$

(b) $a_1 = a_2 = a_n = 0$

$b_k = -\frac{2}{k}$ so $b_k \sin kx = -2\left(\frac{\sin kx}{k}\right).$

$$f(x) \approx (1+\pi) - 2\left(\sin x + \frac{\sin 2x}{2} + \cdots + \frac{\sin nx}{n}\right)$$

3. (a) The space W of continuous functions of the form $a + be^x$ over [0, 1] is spanned by $\mathbf{v}_1 = 1$ and $\mathbf{v}_2 = e^x$.

Let $\mathbf{g}_1 = \mathbf{v}_1 = 1$.

$$\|\mathbf{g}_1\|^2 = \int_0^1 1^2\,dx = 1$$

$$\langle \mathbf{v}_2, \mathbf{g}_1\rangle = \int_0^1 e^x dx = e^x\Big|_0^1 = e - 1$$

$$\begin{aligned} \mathbf{g}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{g}_1\rangle}{\|\mathbf{g}_1\|^2}\mathbf{g}_1 \\ &= e^x - \frac{e-1}{1}(1) \\ &= 1 - e + e^x \end{aligned}$$

$$\begin{aligned} &\|\mathbf{g}_2\|^2 \\ &= \int_0^1 (1-e+e^x)^2\,dx \\ &= \int_0^1 (1 - 2e + e^2 + 2e^x - 2ee^x + e^{2x})dx \\ &= \left(x - 2ex + e^2x + 2e^x - 2e^{x+1} + \frac{1}{2}e^{2x}\right)\Bigg|_0^1 \\ &= -\frac{3}{2} + 2e - \frac{1}{2}e^2 \\ &= -\frac{1}{2}(1-e)(3-e) \end{aligned}$$

$\mathbf{g}_1$ and $\mathbf{g}_2$ are an orthogonal basis for W.
The least squares approximation of x is the orthogonal projection of x on W,

$$\text{proj}_W x = \frac{\langle x, \mathbf{g}_1\rangle}{\|\mathbf{g}_1\|^2}\mathbf{g}_1 + \frac{\langle x, \mathbf{g}_2\rangle}{\|\mathbf{g}_2\|^2}\mathbf{g}_2$$

$$\langle x, \mathbf{g}_1\rangle = \int_0^1 x\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$$

$$\begin{aligned} \langle x, \mathbf{g}_2\rangle &= \int_0^1 x(1-e+e^x)dx \\ &= \frac{3}{2} - \frac{1}{2}e \\ &= \frac{1}{2}(3-e) \end{aligned}$$

$$\begin{aligned} \text{proj}_W x &= \frac{\frac{1}{2}}{1}(1) + \frac{\frac{1}{2}(3-e)}{-\frac{1}{2}(1-e)(3-e)}(1-e+e^x) \\ &= \frac{1}{2} - \frac{1}{1-e}(1-e+e^x) \\ &= \frac{1}{2} - \frac{1-e}{1-e} - \frac{1}{1-e}e^x \\ &= -\frac{1}{2} + \frac{1}{e-1}e^x \end{aligned}$$

(b) Let $\mathbf{f}_1 = x$ and $\mathbf{f}_2 = \text{proj}_W \mathbf{f}_1$, then the mean square error is $\|\mathbf{f}_1 - \mathbf{f}_2\|^2$. Recall that $\mathbf{f}_1 - \text{proj}_W \mathbf{f}_1 = \mathbf{f}_1 - \mathbf{f}_2$ and $\mathbf{f}_2$ are orthogonal.

$$\begin{aligned}\|\mathbf{f}_1 - \mathbf{f}_2\|^2 &= \langle \mathbf{f}_1 - \mathbf{f}_2, \mathbf{f}_1 - \mathbf{f}_2\rangle \\ &= \langle \mathbf{f}_1, \mathbf{f}_1 - \mathbf{f}_2\rangle - \langle \mathbf{f}_2, \mathbf{f}_1 - \mathbf{f}_2\rangle \\ &= \langle \mathbf{f}_1, \mathbf{f}_1 - \mathbf{f}_2\rangle \\ &= \langle \mathbf{f}_1, \mathbf{f}_1\rangle - \langle \mathbf{f}_1, \mathbf{f}_2\rangle \\ &= \|\mathbf{f}_1\|^2 - \langle \mathbf{f}_1, \mathbf{f}_2\rangle\end{aligned}$$

Now decompose $\mathbf{f}_2$ in terms of $\mathbf{g}_1$ and $\mathbf{g}_2$ from part (a).

$$\begin{aligned}&\|\mathbf{f}_1 - \mathbf{f}_2\|^2 \\ &= \|\mathbf{f}_1\|^2 - \frac{\langle \mathbf{f}_1, \mathbf{g}_1\rangle}{\|\mathbf{g}_1\|^2}\langle \mathbf{f}_1, \mathbf{g}_1\rangle - \frac{\langle \mathbf{f}_1, \mathbf{g}_2\rangle}{\|\mathbf{g}_2\|^2}\langle \mathbf{f}_1, \mathbf{g}_2\rangle \\ &= \|\mathbf{f}_1\|^2 - \frac{\langle \mathbf{f}_1, \mathbf{g}_1\rangle^2}{\|\mathbf{g}_1\|^2} - \frac{\langle \mathbf{f}_1, \mathbf{g}_2\rangle^2}{\|\mathbf{g}_2\|^2}\end{aligned}$$

$$\|\mathbf{f}_1\|^2 = \int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

Thus,

$$\begin{aligned}\|\mathbf{f}_1 - \mathbf{f}_2\|^2 &= \frac{1}{3} - \frac{\left(\frac{1}{2}\right)^2}{1} - \frac{\left(\frac{1}{2}(3-e)\right)^2}{-\frac{1}{2}(1-e)(3-e)} \\ &= \frac{1}{3} - \frac{1}{4} + \frac{3-e}{2(1-e)} \\ &= \frac{1}{12} + \frac{2-2e+1+e}{2(1-e)} \\ &= \frac{13}{12} + \frac{1+e}{2(1-e)}\end{aligned}$$

5. **(a)** The space W of continuous functions of the form $a_0 + a_1x + a_2x^2$ over $[-1, 1]$ is spanned by $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$, and $\mathbf{v}_3 = x^2$. Let $\mathbf{g}_1 = \mathbf{v}_1 = 1$.

$$\|\mathbf{g}_1\|^2 = \int_{-1}^1 1^2\,dx = x|_{-1}^1 = 2$$

$$\langle \mathbf{v}_2, \mathbf{g}_1\rangle = \int_{-1}^1 x\,dx = \left.\frac{1}{2}x^2\right|_{-1}^1 = 0$$

$$\mathbf{g}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{g}_1\rangle}{\|\mathbf{g}_1\|^2}\mathbf{g}_1 = \mathbf{v}_2 = x$$

$$\|\mathbf{g}_2\|^2 = \int_{-1}^1 x^2\,dx = \left.\frac{1}{3}x^3\right|_{-1}^1 = \frac{2}{3}$$

$$\langle \mathbf{v}_3, \mathbf{g}_1\rangle = \int_{-1}^1 x^2\,dx = \frac{2}{3}$$

$$\langle \mathbf{v}_3, \mathbf{g}_2\rangle = \int_{-1}^1 x^3\,dx = \left.\frac{1}{4}x^4\right|_{-1}^1 = 0$$

$$\begin{aligned}\mathbf{g}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{g}_1\rangle}{\|\mathbf{g}_1\|^2}\mathbf{g}_1 - \frac{\langle \mathbf{v}_3, \mathbf{g}_2\rangle}{\|\mathbf{g}_2\|^2}\mathbf{g}_2 \\ &= x^2 - \frac{\frac{2}{3}}{2}(1) - 0 \\ &= -\frac{1}{3} + x^2\end{aligned}$$

$\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ is an orthogonal basis for W. The least squares approximation of $\mathbf{f}_1 = \sin \pi x$ is the orthogonal projection of $\mathbf{f}_1$ on W,

$$\text{proj}_W \mathbf{f}_1 = \frac{\langle \mathbf{f}_1, \mathbf{g}_1\rangle}{\|\mathbf{g}_1\|^2}\mathbf{g}_1 + \frac{\langle \mathbf{f}_1, \mathbf{g}_2\rangle}{\|\mathbf{g}_2\|^2}\mathbf{g}_2 + \frac{\langle \mathbf{f}_1, \mathbf{g}_3\rangle}{\|\mathbf{g}_3\|^2}\mathbf{g}_3$$

$$\langle \mathbf{f}_1, \mathbf{g}_1\rangle = \int_{-1}^1 \sin \pi x\,dx = \left.-\frac{1}{\pi}\cos \pi x\right|_{-1}^1 = 0$$

$$\langle \mathbf{f}_1, \mathbf{g}_2\rangle = \int_{-1}^1 x \sin \pi x\,dx = \frac{2}{\pi}$$

$$\langle \mathbf{f}_1, \mathbf{g}_3\rangle = \int_{-1}^1 \left(-\frac{1}{3} + x^2\right)\sin \pi x\,dx = 0$$

$$\text{proj}_W \mathbf{f}_1 = 0\mathbf{g}_1 + \frac{\frac{2}{\pi}}{\frac{2}{3}}x + 0\mathbf{g}_3 = \frac{3}{\pi}x$$

(b) Let $\mathbf{f}_2 = \text{proj}_W \mathbf{f}_1$, then in a clear extension of the process in Exercise 3(b),

$$\begin{aligned}&\|\mathbf{f}_1 - \mathbf{f}_2\|^2 \\ &= \|\mathbf{f}_1\|^2 - \frac{\langle \mathbf{f}_1, \mathbf{g}_1\rangle^2}{\|\mathbf{g}_1\|^2} - \frac{\langle \mathbf{f}_1, \mathbf{g}_2\rangle^2}{\|\mathbf{g}_2\|^2} - \frac{\langle \mathbf{f}_1, \mathbf{g}_3\rangle^2}{\|\mathbf{g}_3\|^2}\end{aligned}$$

$$\|\mathbf{f}_1\|^2 = \int_{-1}^1 \sin^2 \pi x\,dx = 1$$

Thus, the mean square error is

$$\|\mathbf{f}_1 - \mathbf{f}_2\|^2 = 1 - 0 - \frac{\left(\frac{2}{\pi}\right)^2}{\frac{2}{3}} - 0 = 1 - \frac{6}{\pi^2}$$

9. Let $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} dx = 1$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_0^{\pi}\cos kx\,dx = 0$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx \\ &= \frac{1}{\pi}\int_0^{\pi}\sin kx\,dx \\ &= \frac{1}{k\pi}(1-(-1)^k) \end{aligned}$$

So the Fourier series is

$$\frac{1}{2}+\sum_{k=1}^{\infty}\frac{1}{k\pi}(1-(-1)^k)\sin kx.$$

True/False 6.6

(a) False; the area between the graphs is the error, not the mean square error.

(b) True

(c) True

(d) False; $\|1\| = \langle 1, 1\rangle = \int_0^{2\pi} 1^2\,dx = 2\pi \neq 1$

(e) True

Chapter 6 Supplementary Exercises

1. (a) Let $\mathbf{v} = (v_1, v_2, v_3, v_4)$.
$\langle \mathbf{v}, \mathbf{u}_1\rangle = v_1$, $\langle \mathbf{v}, \mathbf{u}_2\rangle = v_2$, $\langle \mathbf{v}, \mathbf{u}_3\rangle = v_3$, $\langle \mathbf{v}, \mathbf{u}_4\rangle = v_4$
If $\langle \mathbf{v}, \mathbf{u}_1\rangle = \langle \mathbf{v}, \mathbf{u}_4\rangle = 0$, then $v_1 = v_4 = 0$ and $\mathbf{v} = (0, v_2, v_3, 0)$. Since the angle θ between **u** and **v** satisfies $\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}$, then **v** making equal angles with $\mathbf{u}_2$ and $\mathbf{u}_3$ means that $v_2 = v_3$. In order for the angle between **v** and $\mathbf{u}_3$ to be defined $\|\mathbf{v}\| \neq 0$. Thus, $\mathbf{v} = (0, a, a, 0)$ with $a \neq 0$.

(b) As in part (a), since $\langle \mathbf{x}, \mathbf{u}_1\rangle = \langle \mathbf{x}, \mathbf{u}_4\rangle = 0$, $x_1 = x_4 = 0$.
Since $\|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1$ and we want $\|\mathbf{x}\| = 1$, then the cosine of the angle between **x** and $\mathbf{u}_2$ is $\cos\theta_2 = \langle \mathbf{x}, \mathbf{u}_2\rangle = x_2$ and, similarly, $\cos\theta_3 = \langle \mathbf{x}, \mathbf{u}_3\rangle = x_3$, so we want $x_2 = 2x_3$, and $\mathbf{x} = \langle 0, x_2, 2x_2, 0\rangle$.

$$\|\mathbf{x}\| = \sqrt{x_2^2 + 4x_2^2} = \sqrt{5x_2^2} = |x_2|\sqrt{5}.$$

If $\|\mathbf{x}\| = 1$, then $x_2 = \pm\frac{1}{\sqrt{5}}$, so

$$\mathbf{x} = \pm\left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right).$$

3. Recall that if $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4\end{bmatrix}$ and $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4\end{bmatrix}$, then $\langle U, V\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$.

(a) If U is a diagonal matrix, then $u_2 = u_3 = 0$ and $\langle U, V\rangle = u_1v_1 + u_4v_4$.
For V to be in the orthogonal complement of the subspace of all diagonal matrices, then it must be the case that $v_1 = v_4 = 0$ and V must have zeros on the main diagonal.

(b) If U is a symmetric matrix, then $u_2 = u_3$ and $\langle U, V\rangle = u_1v_1 + u_2(v_2 + v_3) + u_4v_4$.
Since u_1 and u_4 can take on any values, for V to be in the orthogonal complement of the subspace of all symmetric matrices, it must be the case that $v_1 = v_4 = 0$ and $v_2 = -v_3$, thus V must be skew-symmetric.

5. Let $\mathbf{u} = \left(\sqrt{a_1}, ..., \sqrt{a_n}\right)$ and $\mathbf{v} = \left(\frac{1}{\sqrt{a_1}}, ..., \frac{1}{\sqrt{a_n}}\right)$. By the Cauchy-Schwarz Inequality, $\langle \mathbf{u}\cdot\mathbf{v}\rangle^2 = \underbrace{(1+\cdots+1)^2}_{n\text{ terms}} \le \|\mathbf{u}\|^2\|\mathbf{v}\|^2$ or

$$n^2 \le (a_1 + \cdots + a_1)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right).$$

7. Let $\mathbf{x} = (x_1, x_2, x_3)$.
$\langle \mathbf{x}, \mathbf{u}_1\rangle = x_1 + x_2 - x_3$
$\langle \mathbf{x}, \mathbf{u}_2\rangle = -2x_1 - x_2 + 2x_3$
$\langle \mathbf{x}, \mathbf{u}_3\rangle = -x_1 + x_3$
$\langle \mathbf{x}_1, \mathbf{u}_3\rangle = 0 \Rightarrow -x_1 + x_3 = 0$, so $x_1 = x_3$. Then $\langle \mathbf{x}, \mathbf{u}_1\rangle = x_2$ and $\langle \mathbf{x}, \mathbf{u}_2\rangle = -x_2$, so $x_2 = 0$ and $\mathbf{x} = (x_1, 0, x_1)$. Then

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_1^2} = \sqrt{2x_1^2} = |x_1|\sqrt{2}.$$

If $\|\mathbf{x}\| = 1$ then $x_1 = \pm\frac{1}{\sqrt{2}}$ and the vectors are $\pm\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$.

9. For $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ in R^2, let $\langle \mathbf{u}, \mathbf{v}\rangle = au_1v_1 + bu_2v_2$ be a weighted inner product. If $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (3, -1)$ form an orthonormal set, then

$\|\mathbf{u}\|^2 = a(1)^2 + b(2)^2 = a + 4b = 1$,

$\|\mathbf{v}\|^2 = a(3)^2 + b(-1)^2 = 9a + b = 1$, and

$\langle \mathbf{u}, \mathbf{v}\rangle = a(1)(3) + b(2)(-1) = 3a - 2b = 0$.

This leads to the system $\begin{bmatrix} 1 & 4 \\ 9 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Since $\left[\begin{array}{rr|r} 1 & 4 & 1 \\ 9 & 1 & 1 \\ 3 & -2 & 0 \end{array}\right]$ reduces to $\left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$, the system is inconsistent and there is no such weighted inner product.

11. (a) Let $\mathbf{u}_1 = (k, 0, 0, ..., 0)$, $\mathbf{u}_2 = (0, k, 0, ..., 0)$, ..., $\mathbf{u}_2 = (0, 0, 0, ..., k)$ be the edges of the 'cube' in R^n and $\mathbf{u} = (k, k, k, ..., k)$ be the diagonal.

Then $\|\mathbf{u}_i\| = k$, $\|\mathbf{u}\| = k\sqrt{n}$, and $\langle \mathbf{u}_i, \mathbf{u}\rangle = k^2$, so

$$\cos\theta = \frac{\langle \mathbf{u}_i, \mathbf{u}\rangle}{\|\mathbf{u}_i\|\|\mathbf{u}\|} = \frac{k^2}{k\left(k\sqrt{n}\right)} = \frac{1}{\sqrt{n}}.$$

(b) As n approaches ∞, $\frac{1}{\sqrt{n}}$ approaches 0, so θ approaches $\frac{\pi}{2}$.

13. Recall that $\mathbf{u}$ can be expressed as the linear combination $\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ where $a_i = \langle \mathbf{u}, \mathbf{v}_i\rangle$ for $i = 1, ..., n$. Thus

$$\begin{aligned}\cos^2\alpha_i &= \left(\frac{\langle \mathbf{u}, \mathbf{v}_i\rangle}{\|\mathbf{u}\|\|\mathbf{v}_i\|}\right)^2 \\ &= \left(\frac{a_i}{\|\mathbf{u}\|}\right)^2 \qquad (\|\mathbf{v}_i\| = 1) \\ &= \frac{a_i^2}{a_1^2 + a_2^2 + \cdots + a_n^2}.\end{aligned}$$

Therefore

$$\cos^2\alpha_1 + \cdots + \cos^2\alpha_n = \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{a_1^2 + a_2^2 + \cdots + a_n^2} = 1.$$

15. To show that $(W^\perp)^\perp = W$, we first show that $W \subseteq (W^\perp)^\perp$. If $\mathbf{w}$ is in W, then $\mathbf{w}$ is orthogonal to every vector in $W^\perp$, so that $\mathbf{w}$ is in $(W^\perp)^\perp$. Thus $W \subseteq (W^\perp)^\perp$.

To show that $(W^\perp)^\perp \subseteq W$, let $\mathbf{v}$ be in $(W^\perp)^\perp$. Since $\mathbf{v}$ is in V, we have, by the Projection Theorem, that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^\perp$. By definition, $\langle \mathbf{v}, \mathbf{w}_2\rangle = \langle \mathbf{w}_1, \mathbf{w}_2\rangle = 0$. But

$$\begin{aligned}\langle \mathbf{v}, \mathbf{w}_2\rangle &= \langle \mathbf{w}_1 + \mathbf{w}_2, \mathbf{w}_2\rangle \\ &= \langle \mathbf{w}_1, \mathbf{w}_2\rangle + \langle \mathbf{w}_2, \mathbf{w}_2\rangle \\ &= \langle \mathbf{w}_2, \mathbf{w}_2\rangle\end{aligned}$$

so that $\langle \mathbf{w}_2, \mathbf{w}_2\rangle = 0$. Hence $\mathbf{w}_2 = \mathbf{0}$ and therefore $\mathbf{v} = \mathbf{w}_1$, so that $\mathbf{v}$ is in W. Thus $(W^\perp)^\perp \subseteq W$.

17. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}$,

$A^TA = \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}$,

$A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} 4s + 3 \\ 5s + 2 \end{bmatrix}$

The associated normal system is $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4s + 3 \\ 5s + 2 \end{bmatrix}$.

If the least squares solution is $x_1 = 1$ and $x_2 = 2$, then $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 71 \\ 95 \end{bmatrix} = \begin{bmatrix} 4s + 3 \\ 5s + 2 \end{bmatrix}$.

The resulting equations have solutions $s = 17$ and $s = 18.6$, respectively, so no such value of s exists.

Chapter 7

Diagonalization and Quadratic Forms

Section 7.1

Exercise Set 7.1

1. (b) Since A is orthogonal,

$$A^{-1} = A^T = \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}.$$

3. (a) For the given matrix A,

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A is orthogonal with inverse $A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

(b) For the given matrix A,

$$A^T A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A is orthogonal with inverse

$$A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Note that the given matrix is the standard matrix for a rotation of 45°.

(c) $\|\mathbf{r}_1\| = \sqrt{0^2 + 1^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}} \neq 1$ so the matrix is not orthogonal.

(d) For the given matrix A,

$$A^T A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A is orthogonal with inverse

$$A^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

(e) For the given matrix A,

$$A^T A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A is orthogonal with inverse

$$A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}.$$

(f) $\|\mathbf{r}_2\| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{7}{12}} \neq 1$

The matrix is not orthogonal.

7. (a) $\cos\frac{\pi}{3} = \frac{1}{2},\ \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} -1+3\sqrt{3} \\ \sqrt{3}+3 \end{bmatrix}$$

Thus, $(x', y') = \left(-1+3\sqrt{3}, 3+\sqrt{3}\right).$

(b) Since a rotation matrix is orthogonal, Equation (2) gives
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2}-\sqrt{3} \\ \frac{5\sqrt{3}}{2}+1 \end{bmatrix}.$$
Thus, $(x, y) = \left(\frac{5}{2}-\sqrt{3}, \frac{5}{2}\sqrt{3}+1\right)$.

9. If $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is the standard basis for R^3, and $B' = \{\mathbf{u}_1', \mathbf{u}_2', \mathbf{u}_3'\}$ is the rotated basis, then the transition matrix from B' to B is
$$P = \begin{bmatrix} \cos\frac{\pi}{3} & 0 & \sin\frac{\pi}{3} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{3} & 0 & \cos\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(a) $$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = P^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{2}-\frac{5}{2}\sqrt{3} \\ 2 \\ -\frac{1}{2}\sqrt{3}+\frac{5}{2} \end{bmatrix}$$
Thus
$$(x', y', z') = \left(-\frac{1}{2}-\frac{5}{2}\sqrt{3}, 2, -\frac{1}{2}\sqrt{3}+\frac{5}{2}\right).$$

(b) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2}-\frac{3}{2}\sqrt{3} \\ 6 \\ -\frac{1}{2}\sqrt{3}-\frac{3}{2} \end{bmatrix}$$
Thus $(x, y, z) = \left(\frac{1}{2}-\frac{3}{2}\sqrt{3}, 6, -\frac{1}{2}\sqrt{3}-\frac{3}{2}\right)$.

11. (a) If $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is the standard basis for R^3 and $B' = \{\mathbf{u}_1', \mathbf{u}_2', \mathbf{u}_3'\}$, then
$$[\mathbf{u}_1']_B = \begin{bmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{bmatrix},\ [\mathbf{u}_2']_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and}$$
$$[\mathbf{u}_3']_B = \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix}.$$ Thus the transition matrix from B' to B is $P = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$,
i.e., $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$. Then
$$A = P^{-1} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

(b) With the same notation, $[\mathbf{u}_1']_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,
$[\mathbf{u}_2']_B = \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}$, and $[\mathbf{u}_3']_B = \begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix}$, so the transition matrix from B' to B is
$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \text{ and}$$
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}.$$

13. Let $A = \begin{bmatrix} a+b & b-a \\ a-b & b+a \end{bmatrix}$. Then
$$A^TA = \begin{bmatrix} 2(a^2+b^2) & 0 \\ 0 & 2(a^2+b^2) \end{bmatrix},$$ so a and b must satisfy $a^2+b^2 = \frac{1}{2}$.

17. The row vectors of an orthogonal matrix are an orthonormal set.
$$\|\mathbf{r}_1\|^2 = a^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = a^2+1$$
Thus $a = 0$.
$$\|\mathbf{r}_2\|^2 = b^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 = b^2+\frac{2}{6}$$
Thus $b = \pm\frac{2}{\sqrt{6}}$.
$$\|\mathbf{r}_3\|^2 = c^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = c^2+\frac{2}{3}$$

Thus $c = \pm\frac{1}{\sqrt{3}}$.

The column vectors of an orthogonal matrix are an orthonormal set, from which it is clear that b and c must have opposite signs. Thus the only possibilities are $a = 0$, $b = \frac{2}{\sqrt{6}}$, $c = -\frac{1}{\sqrt{3}}$ or $a = 0$, $b = -\frac{2}{\sqrt{2}}$, $c = \frac{1}{\sqrt{3}}$.

21. (a) Rotations about the origin, reflections about any line through the origin, and any combination of these are rigid operators.

(b) Rotations about the origin, dilations, contractions, reflections about lines through the origin, and combinations of these are angle preserving.

(c) All rigid operators on R^2 are angle preserving. Dilations and contractions are angle preserving operators that are not rigid.

True/False 7.1

(a) False; only square matrices can be orthogonal.

(b) False; the row and column vectors are not unit vectors.

(c) False; only square matrices can be orthogonal.

(d) False; the column vectors must form an orthonormal set.

(e) True; since $A^TA = I$ for an orthogonal matrix A, A must be invertible.

(f) True; a product of orthogonal matrices is orthogonal, so A^2 is orthogonal, hence $\det(A^2) = (\det A)^2 = (\pm 1)^2 = 1$.

(g) True; since $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for an orthogonal matrix.

(h) True; for any nonzero vector $\mathbf{x}$, $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ is a unit vector, so $\left\|A\frac{\mathbf{x}}{\|\mathbf{x}\|}\right\| = 1$. It follows that $\frac{1}{\|\mathbf{x}\|}\|A\mathbf{x}\| = 1$ and $\|A\mathbf{x}\| = \|\mathbf{x}\|$, so $\mathbf{A}$ is orthogonal by Theorem 7.1.3.

Section 7.2

Exercise Set 7.2

1. (a) $\begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-4 \end{vmatrix} = \lambda^2 - 5\lambda$

The characteristic equation is $\lambda^2 - 5\lambda = 0$ and the eigenvalues are $\lambda = 0$ and $\lambda = 5$. Both eigenspaces are one-dimensional.

(b) $\begin{vmatrix} \lambda-1 & 4 & -2 \\ 4 & \lambda-1 & 2 \\ -2 & 2 & \lambda+2 \end{vmatrix} = \lambda^3 - 27\lambda - 54$

$= (\lambda-6)(\lambda+3)^2$

The characteristic equation is $\lambda^3 - 27\lambda - 54 = 0$ and the eigenvalues are $\lambda = 6$ and $\lambda = -3$. The eigenspace for $\lambda = 6$ is one-dimensional; the eigenspace for $\lambda = -3$ is two-dimensional.

(c) $\begin{vmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{vmatrix} = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda-3)$

The characteristic equation is $\lambda^3 - 3\lambda^2 = 0$ and the eigenvalues are $\lambda = 3$ and $\lambda = 0$. The eigenspace for $\lambda = 3$ is one-dimensional; the eigenspace for $\lambda = 0$ is two-dimensional.

(d) $\begin{vmatrix} \lambda-4 & -2 & -2 \\ -2 & \lambda-4 & -2 \\ -2 & -2 & \lambda-4 \end{vmatrix} = \lambda^3 - 12\lambda^2 + 36\lambda - 32$

$= (\lambda-8)(\lambda-2)^2$

The characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$ and the eigenvalues are $\lambda = 8$ and $\lambda = 2$. The eigenspace for $\lambda = 8$ is one-dimensional; the eigenspace for $\lambda = 2$ is two-dimensional.

(e) $\begin{vmatrix} \lambda-4 & -4 & 0 & 0 \\ -4 & \lambda-4 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda^4 - 8\lambda^3 = \lambda^3(\lambda-8)$

The characteristic equation is $\lambda^4 - 8\lambda^3 = 0$ and the eigenvalues are $\lambda = 0$ and $\lambda = 8$. The eigenspace for $\lambda = 0$ is three-dimensional; the eigenspace for $\lambda = 8$ is one-dimensional.

(f) $\begin{vmatrix} \lambda-2 & 1 & 0 & 0 \\ 1 & \lambda-2 & 0 & 0 \\ 0 & 0 & \lambda-2 & 1 \\ 0 & 0 & 1 & \lambda-2 \end{vmatrix}$
$= \lambda^4 - 8\lambda^3 + 22\lambda^2 - 24\lambda + 9$
$= (\lambda-1)^2(\lambda-3)^2$

The characteristic equation is $\lambda^4 - 8\lambda^3 + 22\lambda^2 - 24\lambda + 9 = 0$ and the eigenvalues are $\lambda = 1$ and $\lambda = 3$. Both eigenspaces are two-dimensional.

3. The characteristic equation of A is $\lambda^2 - 13\lambda + 30 = (\lambda-3)(\lambda-10) = 0$.

$\lambda_1 = 3$: A basis for the eigenspace is $\mathbf{x}_1 = \begin{bmatrix} -\frac{2}{\sqrt{3}} \\ 1 \end{bmatrix}$. $\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} -\frac{2}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} \end{bmatrix}$ is an orthonormal basis.

$\lambda_2 = 10$: A basis for the eigenspace is $\mathbf{x}_2 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}$. $\mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{2}{\sqrt{7}} \end{bmatrix}$ is an orthonormal basis. Thus $P = \begin{bmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{bmatrix}$ orthogonally diagonalizes A and

$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix}$.

5. The characteristic equation of A is
$\lambda^3 + 28\lambda^2 - 1175\lambda - 3750$
$= (\lambda-25)(\lambda+3)(\lambda+50)$
$= 0$.

$\lambda_1 = 25$: A basis for the eigenspace is $\mathbf{x}_1 = \begin{bmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$. $\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} -\frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}$ is an orthonormal basis.

$\lambda_2 = -3$: A basis for the eigenspace is $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{v}_2$, since $\|\mathbf{x}_2\| = 1$.

$\lambda_3 = -50$: A basis for the eigenspace is $\mathbf{x}_3 = \begin{bmatrix} \frac{3}{4} \\ 0 \\ 1 \end{bmatrix}$. $\mathbf{v}_3 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|} = \begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix}$ is an orthonormal basis. Thus $P = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$ orthogonally diagonalizes A and

$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -50 \end{bmatrix}$.

7. The characteristic equation of A is $\lambda^3 - 6\lambda^2 + 9\lambda = \lambda(\lambda-3)^2 = 0$.

$\lambda_1 = 0$: A basis for the eigenspace is $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ is an orthonormal basis.

$\lambda_2 = 3$: A basis for the eigenspace is $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. The Gram-Schmidt process gives the orthonormal basis $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$.

Thus, $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$ orthogonally diagonalizes A and

$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

9. The characteristic equation of A is
$\lambda^4 - 1250\lambda^2 + 390,625 = (\lambda+25)^2(\lambda-25)^2$
$= 0$.

$\lambda_1 = -25$: A basis for the eigenspace is

$\mathbf{x}_1 = \begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ -\frac{4}{3} \\ 1 \end{bmatrix}$. The Gram-Schmidt process gives the orthonormal basis $\mathbf{v}_1 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$.

$\lambda_2 = 25$: A basis for the eigenspace is $\mathbf{x}_3 = \begin{bmatrix} \frac{3}{4} \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ \frac{3}{4} \\ 1 \end{bmatrix}$. The Gram-Schmidt process gives the orthonormal basis $\mathbf{v}_3 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$. Thus,

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4] = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

orthogonally diagonalizes A and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & -25 & 0 \\ 0 & 0 & 0 & 25 \end{bmatrix}.$$

15. No, a non-symmetric matrix A can have eigenvalues that are real numbers. For instance, the eigenvalues of the matrix $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ are 3 and -1.

17. If A is an orthogonal matrix then its eigenvalues have absolute value 1, but may be complex. Since the eigenvalues of a symmetric matrix must be real numbers, the only possible eigenvalues for an orthogonal symmetric matrix are 1 and -1.

19. Yes; to diagonalize A, follow the steps on page 399. The Gram-Schmidt process will ensure that columns of P corresponding to the same eigenvalue are an orthonormal set. Since eigenvectors from distinct eigenvalues are orthogonal, this means that P will be an orthogonal matrix. Then since A is orthogonally diagonalizable, it must be symmetric.

True/False 7.2

(a) True; for any square matrix A, both AA^T and A^TA are symmetric, hence orthogonally diagonalizable.

(b) True; since $\mathbf{v}_1$ and $\mathbf{v}_2$ are from distinct eigenspaces of a symmetric matrix, they are orthogonal, so

$$\begin{aligned} \|\mathbf{v}_1 + \mathbf{v}_2\|^2 &= \langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 \rangle \\ &= \langle \mathbf{v}_1, \mathbf{v}_1 \rangle + 2\langle \mathbf{v}_1, \mathbf{v}_2 \rangle + \langle \mathbf{v}_2, \mathbf{v}_2 \rangle \\ &= \|\mathbf{v}_1\|^2 + 0 + \|\mathbf{v}_2\|^2. \end{aligned}$$

(c) False; an orthogonal matrix is not necessarily symmetric.

(d) True; $(P^{-1}AP)^{-1} = P^{-1}A^{-1}P$ since P^{-1} and A^{-1} both exist and since $P^{-1}AP$ is diagonal, so is $(P^{-1}AP)^{-1}$.

(e) True; since $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for an orthogonal matrix.

(f) True; if A is an $n \times n$ orthogonally diagonalizable matrix, then A has an orthonormal set of n eigenvectors, which form a basis for R^n.

(g) True; if A is orthogonally diagonalizable, then A must be symmetric, so the eigenvalues of A will all be real numbers.

Section 7.3

Exercise Set 7.3

1. (a) $3x_1^2 + 7x_2^2 = [x_1 \quad x_2]\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(b) $4x_1^2 - 9x_2^2 - 6x_1x_2 = [x_1 \quad x_2]\begin{bmatrix} 4 & -3 \\ -3 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(c) $9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + x_2x_3$

$= [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & \frac{1}{2} \\ -4 & \frac{1}{2} & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

3. $[x \quad y]\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + 5y^2 - 6xy$

5. $Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2]\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

The characteristic equation of A is $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$, so the eigenvalues are $\lambda = 3, 1$. Orthonormal bases for the eigenspaces are $\lambda = 3$: $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$; $\lambda = 1$: $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

For $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, let $\mathbf{x} = P\mathbf{y}$, then

$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(P^T AP)\mathbf{y} = [y_1 \quad y_2]\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

and $Q = 3y_1^2 + y_2^2$.

7. $Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The characteristic equation of A is $\lambda^3 - 12\lambda^2 + 39\lambda - 28 = (\lambda - 1)(\lambda - 4)(\lambda - 7) = 0$, so the eigenvalues are $\lambda = 1, 4, 7$. Orthonormal bases for the eigenspaces are $\lambda = 1$: $\begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$;

$\lambda = 4$: $\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$; $\lambda = 7$: $\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$.

For $P = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$, let $\mathbf{x} = P\mathbf{y}$, then

$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(P^T AP)\mathbf{y}$

$= [y_1 \quad y_2 \quad y_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and

$Q = y_1^2 + 4y_2^2 + 7y_3^2$.

9. (a) $2x^2 + xy + x - 6y + 2 = 0$ can be written as $2x^2 + 0y^2 + xy + (x - 6y) + 2 = 0$ or

$[x \quad y]\begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + [1 \quad -6]\begin{bmatrix} x \\ y \end{bmatrix} + 2 = 0.$

(b) $y^2 + 7x - 8y - 5 = 0$ can be written as $0x^2 + y^2 + 0xy + (7x - 8y) - 5 = 0$ or

$[x \quad y]\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + [7 \quad -8]\begin{bmatrix} x \\ y \end{bmatrix} - 5 = 0.$

11. (a) $2x^2 + 5y^2 = 20$ is $\frac{x^2}{10} + \frac{y^2}{4} = 1$ which is the equation of an ellipse.

(b) $x^2 - y^2 - 8 = 0$ is $x^2 - y^2 = 8$ or $\frac{x^2}{8} - \frac{y^2}{8} = 1$ which is the equation of a hyperbola.

(c) $7y^2 - 2x = 0$ is $x = \frac{7}{2}y^2$ which is the equation of a parabola.

(d) $x^2 + y^2 - 25 = 0$ is $x^2 + y^2 = 25$ which is the equation of a circle.

13. The equation can be written in matrix form as $\mathbf{x}^T A\mathbf{x} = -8$ where $A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -2$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ respectively. Thus the matrix $P = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$ orthogonally diagonalizes A. Note that det(P) = 1, so P is a rotation matrix. The equation of the conic in the rotated $x'y'$-coordinate system is $[x' \quad y']\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = -8$ which can be written as $2y'^2 - 3x'^2 = 8$; thus the conic is a hyperbola. The angle through which the axes have been rotated is $\theta = \tan^{-1}\left(-\frac{1}{2}\right) \approx -26.6°$.

15. The equation can be written in matrix form as $\mathbf{x}^T A\mathbf{x} = 15$ where $A = \begin{bmatrix} 11 & 12 \\ 12 & 4 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 20$ and $\lambda_2 = -5$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ respectively. Thus the matrix $P = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ orthogonally diagonalizes A. Note that det(P) = 1, so P is a rotation matrix. The equation of the conic in the rotated $x'y'$-coordinate system is $[x' \quad y']\begin{bmatrix} 20 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = 15$ which we can write as $4x'^2 - y'^2 = 3$; thus the conic is a hyperbola. The angle through which the axes have been rotated is $\theta = \tan^{-1}\left(\frac{3}{4}\right) \approx 36.9°$.

17. (a) The eigenvalues of $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ are $\lambda = 1$ and $\lambda = 2$, so the matrix is positive definite.

(b) The eigenvalues of $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ are $\lambda = -1$ and $\lambda = -2$, so the matrix is negative definite.

(c) The eigenvalues of $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ are $\lambda = -1$ and $\lambda = 2$, so the matrix is indefinite.

(d) The eigenvalues of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are $\lambda = 1$ and $\lambda = 0$, so the matrix is positive semidefinite.

(e) The eigenvalues of $\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$ are $\lambda = 0$ and $\lambda = -2$, so the matrix is negative semidefinite.

19. We have $Q = x_1^2 + x_2^2 > 0$ for $(x_1, x_2) \neq (0, 0)$; thus Q is positive definite.

21. We have $Q = (x_1 - x_2)^2 > 0$ for $x_1 \neq x_2$ and $Q = 0$ for $x_1 = x_2$; thus Q is positive semidefinite.

23. We have $x_1^2 - x_2^2 > 0$ for $x_1 \neq 0$, $x_2 = 0$ and $Q < 0$ for $x_1 = 0$, $x_2 \neq 0$; thus Q is indefinite.

25. (a) The eigenvalues of the matrix $A = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$ are $\lambda = 3$ and $\lambda = 7$; thus A is positive definite. Since $|5| = 5$ and $\begin{vmatrix} 5 & -2 \\ -2 & 5 \end{vmatrix} = 21$ are positive, we reach the same conclusion using Theorem 7.3.4.

(b) The eigenvalues of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ are $\lambda = 1$, $\lambda = 3$, and $\lambda = 5$; thus A is positive definite. The determinants of the principal submatrices are $|2| = 2$, $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$, and $\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 15$; thus we reach the same conclusion using Theorem 7.3.4.

27. The quadratic form $Q = 5x_1^2 + x_2^2 + kx_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$ can be expressed in matrix notation as $Q = \mathbf{x}^T A\mathbf{x}$ where $A = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & k \end{bmatrix}$. The determinants of the principal submatrices of A are $|5| = 5$, $\begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} = 1$, and $\begin{vmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & k \end{vmatrix} = k - 2$. Thus Q is positive definite if and only if $k > 2$.

31. **(a)** For each $i = 1, ..., n$ we have

$$\begin{aligned}(x_i - \overline{x})^2 &= x_i^2 - 2x_i\overline{x} + \overline{x}^2 \\ &= x_i^2 - 2x_i \frac{1}{n}\sum_{j=1}^{n} x_j + \frac{1}{n^2}\left(\sum_{j=1}^{n} x_j\right)^2 \\ &= x_i^2 - \frac{2}{n}\sum_{j=1}^{n} x_i x_j + \frac{1}{n^2}\left(\sum_{j=1}^{n} x_j^2 + 2\sum_{j=1}^{n}\sum_{k=j+1}^{n} x_j x_k\right).\end{aligned}$$

Thus in the quadratic form $s_x^2 = \frac{1}{n-1}[(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2]$ the coefficient of x_i^2 is $\frac{1}{n-1}\left[1 - \frac{2}{n} + \frac{1}{n^2}n\right] = \frac{1}{n}$, and the coefficient of $x_i x_j$ for $i \neq j$ is $\frac{1}{n-1}\left[-\frac{2}{n} - \frac{2}{n} + \frac{2}{n^2}n\right] = -\frac{2}{n(n-1)}$. It follows that $s_x^2 = \mathbf{x}^T A\mathbf{x}$ where $A = \begin{bmatrix} \frac{1}{n} & -\frac{1}{n(n-1)} & \cdots & -\frac{1}{n(n-1)} \\ -\frac{1}{n(n-1)} & \frac{1}{n} & \cdots & -\frac{1}{n(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n(n-1)} & -\frac{1}{n(n-1)} & \cdots & \frac{1}{n} \end{bmatrix}$.

(b) We have $s_x^2 = \frac{1}{n-1}[(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2] \geq 0$, and $s_x^2 = 0$ if and only if $x_1 = \overline{x}$, $x_2 = \overline{x}$, ..., $x_n = \overline{x}$, i.e., if and only if $x_1 = x_2 = \cdots = x_n$. Thus s_x^2 is a positive semidefinite form.

33. The eigenvalues of A must be positive and equal to each other. That is, A must have a positive eigenvalue of multiplicity 2.

True/False 7.3

(a) True

(b) False; because of the term $4x_1x_2x_3$.

(c) True; $(x_1 - 3x_2)^2 = x_1^2 - 6x_1x_2 + 9x_2^2$.

(d) True; none of the eigenvalues will be 0.

(e) False; a symmetric matrix can be positive semidefinite or negative semidefinite.

(f) True

(g) True; $\mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2$

(h) True

(i) False; this is only true for symmetric matrices.

(j) True

(k) False; there will be no cross product terms if $a_{ij} = -a_{ji}$ for all $i \neq j$.

(l) False; if $c < 0$, $\mathbf{x}^T A\mathbf{x} = c$ has no graph.

Section 7.4

Exercise Set 7.4

1. The quadratic form $5x^2 - y^2$ can be written in matrix notation as $\mathbf{x}^T A\mathbf{x}$ where $A = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -1$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus the constrained maximum is 5 occurring at $(x, y) = (\pm 1, 0)$, and the constrained minimum is -1 occurring at $(x, y) = (0, \pm 1)$.

3. The quadratic form $3x^2 + 7y^2$ can be written in matrix notation as $\mathbf{x}^T A\mathbf{x}$ where $A = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 7$ and $\lambda_2 = 3$, with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus the constrained maximum is 7 occurring at $(x, y) = (0, \pm 1)$, and the constrained minimum is 3 occurring at $(x, y) = (\pm 1, 0)$.

5. The quadratic form $9x^2 + 4y^2 + 3z^2$ can be expressed as $\mathbf{x}^T A\mathbf{x}$ where $A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 9$, $\lambda_2 = 4$, $\lambda_3 = 3$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Thus the constrained maximum is 9 occurring at $(x, y, z) = (\pm 1, 0, 0)$, and the constrained minimum is 3 occurring at $(x, y, z) = (0, 0, \pm 1)$.

7. Rewrite the constraint as $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 1$, then let $x = 2x_1$ and $y = \sqrt{2}y_1$. The problem is now to maximize $z = xy = 2\sqrt{2}x_1 y_1$ subject to $x_1^2 + y_1^2 = 1$. Write

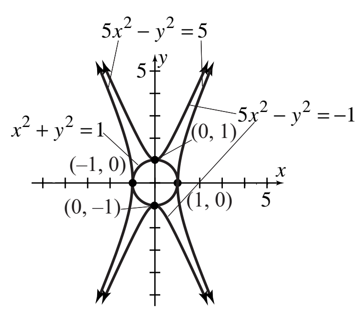
$$z = \mathbf{x}^T A\mathbf{x} = [x_1 \quad y_1]\begin{bmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Since $\begin{vmatrix} \lambda & -\sqrt{2} \\ -\sqrt{2} & \lambda \end{vmatrix} = \lambda^2 - 2 = \left(\lambda + \sqrt{2}\right)\left(\lambda - \sqrt{2}\right)$, the largest eigenvalue of A is $\sqrt{2}$, with corresponding positive unit eigenvector $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Thus the maximum value is $z = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$ occurs when $x = 2x_1 = \sqrt{2}$, $y = \sqrt{2}y_1 = 1$ or $x = -\sqrt{2}$, $y = -1$. The smallest eigenvalue of A is $-\sqrt{2}$, with corresponding unit eigenvectors $\pm\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Thus the minimum value is $z = -\sqrt{2}$ which occurs at $(x, y) = \left(-\sqrt{2}, 1\right)$ and $\left(\sqrt{2}, -1\right)$.

9.



13. The first partial derivatives of f are $f_x(x, y) = 3x^2 - 3y$ and $f_y(x, y) = -3x - 3y^2$. To find the critical points we set f_x and f_y

equal to zero. This yields the equations $y = x^2$ and $x = -y^2$. From this we conclude that $y = y^4$ and so $y = 0$ or $y = 1$. The corresponding values of x are $x = 0$ and $x = -1$ respectively. Thus there are two critical points: (0, 0) and (−1, 1).
The Hessian matrix is
$$H(x, y) = \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & -6y \end{bmatrix}.$$
The eigenvalues of $H(0, 0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$ are $\lambda = \pm 3$; this matrix is indefinite and so f has a saddle point at (0, 0). The eigenvalues of $H(-1, 1) = \begin{bmatrix} -6 & -3 \\ -3 & -6 \end{bmatrix}$ are $\lambda = -3$ and $\lambda = -9$; this matrix is negative definite and so f has a relative maximum at (−1, 1).

15. The first partial derivatives of f are $f_x(x, y) = 2x - 2xy$ and $f_y(x, y) = 4y - x^2$. To find the critical points we set f_x and f_y equal to zero. This yields the equations $2x(1 - y) = 0$ and $y = \frac{1}{4}x^2$. From the first, we conclude that $x = 0$ or $y = 1$. Thus there are three critical points: (0, 0), (2, 1), and (−2, 1).
The Hessian matrix is
$$H(x, y) = \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix} = \begin{bmatrix} 2-2y & -2x \\ -2x & 4 \end{bmatrix}.$$
The eigenvalues of the matrix $H(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ are $\lambda = 2$ and $\lambda = 4$; this matrix is positive definite and so f has a relative minimum at (0, 0).
The eigenvalues of $H(2, 1) = \begin{bmatrix} 0 & -4 \\ -4 & 4 \end{bmatrix}$ are $\lambda = 2 \pm 2\sqrt{5}$. One of these is positive and one is negative; thus this matrix is indefinite and f has a saddle point at (2, 1). Similarly, the eigenvalues of $H(-2, 1) = \begin{bmatrix} 0 & 4 \\ 4 & 4 \end{bmatrix}$ are $\lambda = 2 \pm 2\sqrt{5}$; thus f has a saddle point at (−2, 1).

17. The problem is to maximize $z = 4xy$ subject to $x^2 + 25y^2 = 25$, or $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{1}\right)^2 = 1$. Let $x = 5x_1$ and $y = y_1$, so that the problem is to maximize $z = 20x_1y_1$ subject to $\|(x_1, y_1)\| = 1$.
Write $z = \mathbf{x}^T A\mathbf{x} = [x_1 \quad y_1]\begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$.
$$\begin{vmatrix} \lambda & -10 \\ -10 & \lambda \end{vmatrix} = \lambda^2 - 100 = (\lambda + 10)(\lambda - 10).$$
The largest eigenvalue of A is $\lambda = 10$ which has positive unit eigenvector $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. Thus the maximum value of $z = 20\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 10$ which occurs when $x = 5x_1 = \frac{5}{\sqrt{2}}$ and $y = y_1 = \frac{1}{\sqrt{2}}$, which are the corner points of the rectangle.

21. If $\mathbf{x}$ is a unit eigenvector corresponding to λ, then $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(\lambda\mathbf{x}) = \lambda(\mathbf{x}^T\mathbf{x}) = \lambda(1) = \lambda$.

True/False 7.4

(a) False; if the only critical point of the quadratic form is a saddle point, then it will have neither a maximum nor a minimum value.

(b) True

(c) True

(d) False; the Hessian form of the second derivative test is inconclusive.

(e) True; if $\det(A) < 0$, then A will have negative eigenvalues.

Section 7.5

Exercise Set 7.5

1. $A^* = \begin{bmatrix} -2i & 4 & 5-i \\ 1+i & 3-i & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & i & 2-3i \\ -i & -3 & 1 \\ 2+3i & 1 & 2 \end{bmatrix}$

5. **(a)** A is not Hermitian since $a_{31} \neq \overline{a}_{13}$.

(b) A is not Hermitian since a_{22} is not real.

9. The following computations show that the row vectors of A are orthonormal:

$$\|\mathbf{r}_1\| = \sqrt{\left|\frac{3}{5}\right|^2 + \left|\frac{4}{5}i\right|^2} = \sqrt{\frac{9}{25}+\frac{16}{25}} = 1$$

$$\|\mathbf{r}_2\| = \sqrt{\left|-\frac{4}{5}\right|^2 + \left|\frac{3}{5}i\right|^2} = \sqrt{\frac{16}{25}+\frac{9}{25}} = 1$$

$$\begin{aligned}\mathbf{r}_1 \cdot \mathbf{r}_2 &= \left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) + \left(\frac{4}{5}i\right)\left(-\frac{3}{5}i\right) \\ &= -\frac{12}{5} + \frac{12}{5} \\ &= 0\end{aligned}$$

Thus A is unitary, and $A^{-1} = A^* = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5}i & -\frac{3}{5}i \end{bmatrix}$.

11. The following computations show that the column vectors of A are orthonormal:

$$\begin{aligned}\|\mathbf{c}_1\| &= \sqrt{\left|\frac{1}{2\sqrt{2}}\left(\sqrt{3}+i\right)\right|^2 + \left|\frac{1}{2\sqrt{2}}\left(1+i\sqrt{3}\right)\right|^2} \\ &= \sqrt{\frac{4}{8}+\frac{4}{8}} \\ &= 1\end{aligned}$$

$$\begin{aligned}\|\mathbf{c}_2\| &= \sqrt{\left|\frac{1}{2\sqrt{2}}\left(1-i\sqrt{3}\right)\right|^2 + \left|\frac{1}{2\sqrt{2}}\left(i-\sqrt{3}\right)\right|^2} \\ &= \sqrt{\frac{4}{8}+\frac{4}{8}} \\ &= 1\end{aligned}$$

$$\mathbf{c}_1 \cdot \mathbf{c}_2 = \frac{1}{2\sqrt{2}}\left(\sqrt{3}+i\right)\frac{1}{2\sqrt{2}}\left(1+i\sqrt{3}\right) + \frac{1}{2\sqrt{2}}\left(1+i\sqrt{3}\right)\frac{1}{2\sqrt{2}}\left(-i-\sqrt{3}\right) = 0$$

Thus A is unitary, and $A^{-1} = A^* = \begin{bmatrix} \frac{1}{2\sqrt{2}}\left(\sqrt{3}-i\right) & \frac{1}{2\sqrt{2}}\left(1-i\sqrt{3}\right) \\ \frac{1}{2\sqrt{2}}\left(1+i\sqrt{3}\right) & \frac{1}{2\sqrt{2}}\left(-i-\sqrt{3}\right) \end{bmatrix}$.

13. The characteristic polynomial of A is $\lambda^2 - 9\lambda + 18 = (\lambda-3)(\lambda-6)$; thus the eigenvalues of A are $\lambda = 3$ and $\lambda = 6$. The augmented matrix of the system $(3I - A)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{cc|c} -1 & -1+i & 0 \\ -1-i & -2 & 0 \end{array}\right]$, which reduces to $\left[\begin{array}{cc|c} 1 & 1-i & 0 \\ 0 & 0 & 0 \end{array}\right]$. Thus $\mathbf{v}_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 3$, and $\mathbf{p}_1 = \begin{bmatrix} \frac{-1+i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ is a unit eigenvector. A similar computation shows that $\mathbf{p}_2 = \begin{bmatrix} \frac{1-i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$ is a unit eigenvector corresponding to λ = 6. The vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ are orthogonal, and the unitary matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2]$ diagonalizes the matrix *A*:

$$\begin{aligned} P^*AP &= \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 4 & 1-i \\ 1+i & 5 \end{bmatrix} \begin{bmatrix} \frac{-1+i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix} \end{aligned}$$

15. The characteristic polynomial of *A* is $\lambda^2 - 10\lambda + 16 = (\lambda - 2)(\lambda - 8);$ thus the eigenvalues of *A* are λ = 2 and λ = 8. The augmented matrix of the system $(2I - A)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{cc|c} -4 & -2-2i & 0 \\ -2+2i & -2 & 0 \end{array}\right]$, which reduces to $\left[\begin{array}{cc|c} 2 & 1+i & 0 \\ 0 & 0 & 0 \end{array}\right]$. Thus $\mathbf{v}_1 = \begin{bmatrix} -1-i \\ 2 \end{bmatrix}$ is a basis for the eigenspace corresponding to λ = 2, and $\mathbf{p}_1 = \begin{bmatrix} \frac{-1-i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$ is a unit eigenvector. A similar computation shows that $\mathbf{p}_2 = \begin{bmatrix} \frac{1+i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ is a unit eigenvector corresponding to λ = 8. The vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ are orthogonal, and the unitary matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2]$ diagonalizes the matrix *A*:

$$\begin{aligned} P^*AP &= \begin{bmatrix} \frac{-1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 6 & 2+2i \\ 2-2i & 4 \end{bmatrix} \begin{bmatrix} \frac{-1-i}{\sqrt{6}} & \frac{1+i}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \end{aligned}$$

17. The characteristic polynomial of *A* is $(\lambda - 5)(\lambda^2 + \lambda - 2) = (\lambda + 2)(\lambda - 1)(\lambda - 5);$ thus the eigenvalues of *A* are $\lambda_1 = -2,$ $\lambda_2 = 1,$ and $\lambda_3 = 5.$ The augmented matrix of the system $(-2I - A)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{ccc|c} -7 & 0 & 0 & 0 \\ 0 & -1 & 1-i & 0 \\ 0 & 1+i & -2 & 0 \end{array}\right]$, which can be reduced to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Thus $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1-i \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = -2,$ and $\mathbf{p}_1 = \begin{bmatrix} 0 \\ \frac{1-i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ is a unit eigenvector. Similar computations show that $\mathbf{p}_2 = \begin{bmatrix} 0 \\ \frac{-1+i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$ is a unit eigenvector corresponding to $\lambda_2 = 1,$ and $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a unit eigenvector corresponding to $\lambda_3 = 5.$ The vectors $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ form an orthogonal set, and the unitary matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3]$ diagonalizes the matrix *A*:

$$P^*AP = \begin{bmatrix} 0 & \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1-i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \frac{1-i}{\sqrt{3}} & \frac{-1+i}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Note: The matrix P is not unique. It depends on the particular choice of the unit eigenvectors. This is just one possibility.

19. $A = \begin{bmatrix} 0 & i & 2-3i \\ i & 0 & 1 \\ -2-3i & -1 & 4i \end{bmatrix}$

21. **(a)** $a_{21} = -i \neq i = -\overline{a}_{12}$ and $a_{31} \neq -\overline{a}_{13}$

(b) $a_{11} \neq -\overline{a}_{11}$, $a_{31} \neq -\overline{a}_{13}$, and $a_{32} \neq -\overline{a}_{23}$

23. The characteristic polynomial of $A = \begin{bmatrix} 0 & -1+i \\ 1+i & i \end{bmatrix}$ is $\lambda^2 - i\lambda + 2 = (\lambda - 2i)(\lambda + i)$; thus the eigenvalues of A are $\lambda = 2i$ and $\lambda = -i$.

27. We have $A^*A = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta} & -ie^{-i\theta} \\ e^{i\theta} & ie^{i\theta} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\theta} & e^{-i\theta} \\ ie^{i\theta} & -ie^{-i\theta} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1+1 & e^{-2i\theta} - e^{-2i\theta} \\ e^{2i\theta} - e^{2i\theta} & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; thus $A^* = A^{-1}$ and A is unitary.

29. **(a)** If $B = \frac{1}{2}(A + A^*)$, then
$$\begin{aligned} B^* &= \frac{1}{2}(A + A^*)^* \\ &= \frac{1}{2}(A^* + A^{**}) \\ &= \frac{1}{2}(A^* + A) \\ &= B. \end{aligned}$$
Similarly, $C^* = C$.

(b) We have $B + iC = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) = A$ and $B - iC = \frac{1}{2}(A + A^*) - \frac{1}{2}(A - A^*) = A^*$.

(c)
$$\begin{aligned} AA^* &= (B + iC)(B - iC) \\ &= B^2 - iBC + iCB + C^2 \end{aligned}$$
and $A^*A = B^2 + iBC - iCB + C^2$. Thus $AA^* = A^*A$ if and only if $-iBC + iCB = iBC - iCB$, or $2iCB = 2iBC$. Thus A is normal if and only if B and C commute i.e., $CB = BC$.

31. If A is unitary, then $A^{-1} = A^*$ and so $(A^*)^{-1} = (A^{-1})^* = (A^*)^*$; thus A^* is also unitary.

33. A unitary matrix A has the property that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all x in C^n. Thus if A is unitary and $A\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$, we must have $|\lambda|\|\mathbf{x}\| = \|A\mathbf{x}\| = \|\mathbf{x}\|$ and so $|\lambda| = 1$.

35. If $H = I - 2\mathbf{uu}^*$, then
$H^* = (I - 2\mathbf{uu}^*)^* = I^* - 2\mathbf{u}^{**}\mathbf{u}^* = I - 2\mathbf{uu}^* = H$; thus H is Hermitian.

$$\begin{aligned} HH^* &= (I - 2\mathbf{uu}^*)(I - 2\mathbf{uu}^*) \\ &= I - 2\mathbf{uu}^* - 2\mathbf{uu}^* + 4\mathbf{uu}^*\mathbf{uu}^* \\ &= I - 4\mathbf{uu}^* + 4\mathbf{u}\|\mathbf{u}\|^2\mathbf{u}^* \\ &= I \end{aligned}$$

and so H is unitary.

37. $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ is both Hermitian and unitary.

39. If $P = \mathbf{uu}^*$, then
$P\mathbf{x} = (\mathbf{uu}^*)\mathbf{x} = (\mathbf{u}\overline{\mathbf{u}}^T)\mathbf{x} = \mathbf{u}(\overline{\mathbf{u}}^T\mathbf{x}) = (\mathbf{x} \cdot \mathbf{u})\mathbf{u}$. Thus multiplication of $\mathbf{x}$ by P corresponds to $\|\mathbf{u}\|^2$ times the orthogonal projection of $\mathbf{x}$ onto $W = \text{span}\{\mathbf{u}\}$. If $\|\mathbf{u}\| = 1$, then multiplication of $\mathbf{x}$ by $H = I - 2\mathbf{uu}^*$ corresponds to reflection of $\mathbf{x}$ about the hyperplane $\mathbf{u}^\perp$.

True/False 7.5

(a) False; $i \neq \overline{i}$.

(b) False; for $\mathbf{r}_1 = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{6}} & \frac{i}{\sqrt{3}} \end{bmatrix}$ and
$\mathbf{r}_2 = \begin{bmatrix} 0 & -\frac{i}{\sqrt{6}} & \frac{i}{\sqrt{3}} \end{bmatrix}$,

$$\begin{aligned} \mathbf{r}_1 \cdot \mathbf{r}_2 &= -\frac{i}{\sqrt{2}}(\overline{0}) + \frac{i}{\sqrt{6}}\left(\overline{-\frac{i}{\sqrt{6}}}\right) + \frac{i}{\sqrt{3}}\left(\overline{\frac{i}{\sqrt{3}}}\right) \\ &= 0 + \left(\frac{i}{\sqrt{6}}\right)^2 - \left(\frac{i}{\sqrt{3}}\right)^2 \\ &= -\frac{1}{6} + \frac{1}{3} \\ &= \frac{1}{6} \end{aligned}$$

Thus the row vectors do not form an orthonormal set and the matrix is not unitary.

(c) True; if A is unitary, so $A^{-1} = A^*$, then $(A^*)^{-1} = A = (A^*)^*$.

(d) False; normal matrices that are not Hermitian are also unitarily diagonalizable.

(e) False; if A is skew-Hermitian, then $(A^2)^* = (A^*)(A^*) = (-A)(-A) = A^2 \neq -A^2$.

Chapter 7 Supplementary Exercises

1. (a) For $A = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$, $A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so

$$A^{-1} = A^T = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

(b) For $A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$,

$A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so

$$A^{-1} = A^T = \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}.$$

5. The characteristic equation of A is $\lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda - 2)(\lambda - 1)$, so the eigenvalues are $\lambda = 0, 2, 1$. Orthogonal bases for the eigenspaces are $\lambda = 0$: $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$; $\lambda = 2$: $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$; $\lambda = 1$: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Thus $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ orthogonally diagonalizes A, and $P^TAP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

7. In matrix form, the quadratic form is
$\mathbf{x}^TA\mathbf{x} = [x_1 \quad x_2]\begin{bmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - 5\lambda + \frac{7}{4} = 0$

which has solutions $\lambda = \frac{5 \pm 3\sqrt{2}}{2}$ or $\lambda \approx 4.62, 0.38$. Since both eigenvalues of A are positive, the quadratic form is positive definite.

9. **(a)** $y - x^2 = 0$ or $y = x^2$ represents a parabola.

(b) $3x - 11y^2 = 0$ or $x = \frac{11}{3}y^2$ represents a parabola.

Chapter 8

Linear Transformations

Section 8.1

Exercise Set 8.1

1. $T(-\mathbf{u}) = \|-\mathbf{u}\| = \|\mathbf{u}\| = T(\mathbf{u}) \neq -T(\mathbf{u}),$ so the function is not a linear transformation.

3. $T(kA) = (kA)B = k(AB) = kT(A)$

$$\begin{aligned}T(A_1 + A_2) &= (A_1 + A)B \\ &= A_1B + A_2B \\ &= T(A_1) + T(A_2)\end{aligned}$$

Thus T is a linear transformation.

5. $F(kA) = (kA)^T = kA^T = kF(A)$

$F(A+B) = (A+B)^T = A^T + B^T = F(A) + F(B)$

Thus F is a linear transformation.

7. Let $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2.$

(a)
$$\begin{aligned}T(kp(x)) &= ka_0 + ka_1(x+1) + ka_2(x+1)^2 \\ &= kT(p(x))\end{aligned}$$

$$\begin{aligned}T(p(x)+q(x)) &= a_0 + b_0 + (a_1+b_1)(x+1) + (a_2+b_2)(x+1)^2 \\ &= a_0 + a_1(x+1) + a_2(x+1)^2 + b_0 + b_1(x+1) + b_2(x+1)^2 \\ &= T(p(x)) + T(q(x))\end{aligned}$$

Thus T is a linear transformation.

(b)
$$\begin{aligned}T(kp(x)) &= T(ka_0 + ka_1x + ka_2x^2) \\ &= (ka_0+1) + (ka_1+1)x + (ka_2+1)x^2 \\ &\neq kT(p(x))\end{aligned}$$

T is not a linear transformation.

9. For $\mathbf{x} = (x_1, x_2) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2,$ we have $(x_1, x_2) = c_1(1, 1) + c_2(1, 0) = (c_1 + c_2, c_1)$ or

$$\begin{aligned}c_1 + c_2 &= x_1 \\ c_1 \quad\;\; &= x_2\end{aligned}$$

which has the solution $c_1 = x_2,\ c_2 = x_1 - x_2.$

$$\begin{aligned}(x_1, x_2) &= x_2(1, 1) + (x_1 - x_2)(1, 0) \\ &= x_2\mathbf{v}_1 + (x_1 - x_2)\mathbf{v}_2\end{aligned}$$

and

$$\begin{aligned}T(x_1, x_2) &= x_2T(\mathbf{v}_1) + (x_1 - x_2)T(\mathbf{v}_2) \\ &= x_2(1, -2) + (x_1 - x_2)(-4, 1) \\ &= (-4x_1 + 5x_2,\ x_1 - 3x_2)\end{aligned}$$

$T(5, -3) = (-20 - 15, 5 + 9) = (-35, 14).$

11. For $\mathbf{x} = (x_1, x_2, x_3) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, we have

$$(x_1, x_2, x_3) = c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0)$$
$$= (c_1 + c_2 + c_3, c_1 + c_2, c_1)$$

$$\begin{aligned} c_1 + c_2 + c_3 &= x_1 \\ \text{or } c_1 + c_2 \quad &= x_2 \\ c_1 \quad &= x_3 \end{aligned}$$ which has the solution

$c_1 = x_3$, $c_2 = x_2 - x_3$,
$c_3 = x_1 - (x_2 - x_3) - x_3 = x_1 - x_2$.

$$(x_1, x_2, x_3) = x_3\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_1 - x_2)\mathbf{v}_3$$
$$\begin{aligned} &T(x_1, x_2, x_3) \\ &= x_3T(\mathbf{v}_1) + (x_2 - x_3)T(\mathbf{v}_2) + (x_1 - x_2)T(\mathbf{v}_3) \\ &= x_3(2, -1, 4) + (x_2 - x_3)(3, 0, 1) \\ &\qquad + (x_1 - x_2)(-1, 5, 1) \\ &= (-x_1 + 4x_2 - x_3, 5x_1 - 5x_2 - x_3, x_1 + 3x_3) \end{aligned}$$
$$\begin{aligned} T(2, 4, -1) &= (-2 + 16 + 1, 10 - 20 + 1, 2 - 3) \\ &= (15, -9, -1) \end{aligned}$$

13. $$\begin{aligned} &T(2\mathbf{v}_1 - 3\mathbf{v}_2 + 4\mathbf{v}_3) \\ &= 2T(\mathbf{v}_1) - 3T(\mathbf{v}_2) + 4T(\mathbf{v}_3) \\ &= (2, -2, 4) - (0, 9, 6) + (-12, 4, 8) \\ &= (-10, -7, 6) \end{aligned}$$

15. (a) $T(5, 10) = (10 - 10, -40 + 40) = (0, 0)$ so $(5, 10)$ is in $\ker(T)$.

(b) $T(3, 2) = (6 - 2, -24 + 8) = (4, -16)$ so $(3, 2)$ is not in $\ker(T)$.

(c) $T(1, 1) = (2 - 1, -8 + 4) = (1, -4)$ so $(1, 1)$ is not in $\ker(T)$.

17. (a) $T(3, -8, 2, 0)$
$= (12 - 8 - 4, 6 - 8 + 2, 18 - 18)$
$= (0, 0, 0)$
so $(3, -8, 2, 0)$ is in $\ker(T)$.

(b) $T(0, 0, 0, 1)$
$= (0 + 0 + 0 - 3, 0 + 0 + 0 - 4, 0 - 0 + 9)$
$= (-3, -4, 9)$
so $(0, 0, 0, 1)$ is not in $\ker(T)$.

(c) $$\begin{aligned} T(0, -4, 1, 0) &= (-4 - 2, -4 + 1, -9) \\ &= (-6, -3, -9) \end{aligned}$$
so $(0, -4, 1, 0)$ is not in $\ker(T)$.

19. (a) Since $x + x^2 = x(1 + x)$, $x + x^2$ is $R(T)$.

(b), (c) Neither $1 + x$ nor $3 - x^2$ can be expressed as $xp(x)$ with $p(x)$ in P_2, so neither are in $R(T)$.

21. (a) Since $-8x + 4y = -4(2x - y)$ and $2x - y$ can assume any real value, $R(T)$ is the set of all vectors $(x, -4x)$ or the line $y = -4x$. The vector $(1, -4)$ is a basis for this space.

(b) T can be expressed as a matrix operator from R^4 to R^3 with the matrix $A = \begin{bmatrix} 4 & 1 & -2 & -3 \\ 2 & 1 & 1 & -4 \\ 6 & 0 & -9 & 9 \end{bmatrix}$. A basis for $R(T)$ is a basis for the column space of A. A row echelon form of A is $B = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The columns of A corresponding to the columns of B containing leading 1s are $\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ -4 \\ 9 \end{bmatrix}$. Thus, $(4, 2, 6)$, $(1, 1, 0)$, and $(-3, -4, 9)$ form a basis for $R(T)$.

(c) $R(T)$ is the set of all polynomials in P^3 with constant term 0. Thus, $\{x, x^2, x^3\}$ is a basis for $R(T)$.

23. (a) A row echelon form of A is $\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -\frac{19}{11} \\ 0 & 0 & 0 \end{bmatrix}$, thus columns 1 and 2 of A, $\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$ form a basis for the column space of A, which is the range of T.

(b) A reduces to $\begin{bmatrix} 1 & 0 & \frac{14}{11} \\ 0 & 1 & -\frac{19}{11} \\ 0 & 0 & 0 \end{bmatrix}$, so a general solution of $A\mathbf{x} = \mathbf{0}$ is $x_1 = -\frac{14}{11}s$, $x_2 = \frac{19}{11}s$, $x_3 = s$ or $x_1 = -14t$, $x_2 = 19t$, $x_3 = 11t$, so

$\begin{bmatrix}-14\\19\\11\end{bmatrix}$ is a basis for the null space of A, which is ker(T).

(c) $R(T)$ is two-dimensional, so rank(T) = 2. ker(T) is one-dimensional, so nullity (T) = 1.

(d) The column space of A is two-dimensional, so rank(A) = 2. The null space of A is one-dimensional, so nullity(A) = 1.

25. (a) A row echelon form of A is $\begin{bmatrix}1 & 2 & 3 & 0\\0 & 1 & 1 & -\frac{2}{7}\end{bmatrix}$, thus columns 1 and 2 of A, $\begin{bmatrix}4\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$ form a basis for the column space of A, which is the range of T.

(b) A reduces to $\begin{bmatrix}1 & 0 & 1 & \frac{4}{7}\\0 & 1 & 1 & -\frac{2}{7}\end{bmatrix}$, so a general solution of $A\mathbf{x} = \mathbf{0}$ is $x_1 = -s - \frac{4}{7}t$, $x_2 = -s + \frac{2}{7}t$, $x_3 = s$, $x_4 = t$ or $x_1 = -s - 4r$, $x_2 = -s + 2r$, $x_3 = s$, $x_4 = 7r$, so $\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-4\\2\\0\\7\end{bmatrix}$ form a basis for the null space of A, which is ker(T).

(c) Both $R(T)$ and ker(T) are two-dimensional, so rank(T) = nullity(T) = 2.

(d) Both the column space and the null space of A are two-dimensional, so rank(A) = nullity(A) = 2.

27. (a) The kernel is the y-axis, the range is the entire xz-plane.

(b) The kernel is the x-axis, the range is the entire yz-plane.

(c) The kernel is the line through the origin perpendicular to the plane $y = x$, the range is the entire plane $y = x$.

29. (a) nullity(T) = 5 − rank(T) = 2

(b) $\dim(P_4) = 5$, so nullity(T) = 5 − rank(T) = 4

(c) Since $R(T) = R^3$, T has rank 3. nullity(T) = 6 − rank(T) = 3

(d) nullity(T) = 4 − rank(T) = 1

31. (a) nullity(A) = 7 − rank(A) = 3, so the solution space of $A\mathbf{x} = \mathbf{0}$ has dimension 3.

(b) No, since rank(A) = 4 while R^5 has dimension 5.

33. $R(T)$ must be a subspace of R^3, thus the possibilities are a line through the origin, a plane through the origin, the origin only, or all of R^3.

35. (b) No;
$$\begin{aligned}F(kx, ky) &= (a_1k^2x^2 + b_1k^2y^2, a_2k^2x^2 + b_2k^2y^2)\\ &= k^2F(x, y)\\ &\neq kF(x, y)\end{aligned}$$

37. Let $\mathbf{w} = c\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ be any vector in V. Then
$$\begin{aligned}T(\mathbf{w}) &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n)\\ &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n\\ &= \mathbf{w}\end{aligned}$$
Since $\mathbf{w}$ was an arbitrary vector in V, T must be the identity operator.

41. If $p'(x) = 0$, then $p(x)$ is a constant, so ker(D) consists of all constant polynomials.

43. (a) The fourth derivative of any polynomial of degree 3 or less is 0, so $T(f(x)) = f^{(4)}(x)$ has $\ker(T) = P_3$.

(b) By similar reasoning, $T(f(x)) = f^{(n+1)}(x)$ has $\ker(T) = P_n$.

True/False 8.1

(a) True; $c_1 = k$, $c_2 = 0$ gives the homogeneity property and $c_1 = c_2 = 1$ gives the additivity property.

(b) False; every linear transformation will have $T(-\mathbf{v}) = -T(\mathbf{v})$.

(c) True; only the zero transformation has this property.

(d) False; $T(\mathbf{0}) = \mathbf{v}_0 + \mathbf{0} = \mathbf{v}_0 \neq 0,$ so T is not a linear transformation.

(e) True

(f) True

(g) False; T does not necessarily have rank 4.

(h) False; $\det(A + B) \neq \det(A) + \det(B)$ in general.

(i) False; nullity(T) = rank(T) = 2

Section 8.2

Exercise Set 8.2

1. (a) By inspection, $\ker(T) = \{\mathbf{0}\}$, so T is one-to-one.

(b) $T(x, y) = 0$ if $2x + 3y = 0$ or $x = -\frac{3}{2}y$ so $\ker(T) = \left\{k\left(-\frac{3}{2}, 1\right)\right\}$ and T is not one-to-one.

(c) (x, y) is in $\ker(T)$ only if $x + y = 0$ and $x - y = 0$, so $\ker(T) = \{\mathbf{0}\}$ and T is one-to-one.

(d) By inspection, $\ker(T) = \{\mathbf{0}\}$, so T is one-to-one.

(e) (x, y) is in $\ker(T)$ if $x - y = 0$ or $x = y$, so $\ker(T) = \{k(1, 1)\}$ and T is not one-to-one.

(f) (x, y, z) is in $\ker(T)$ if both $x + y + z = 0$ and $x - y - z = 0$, which is $x = 0$ and $y + z = 0$. Thus, $\ker(T) = \{k(0, 1, -1)\}$ and T is not one-to-one.

3. (a) By inspection, A reduces to $\begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, so nullity$(A) = 1$ and multiplication by A is not one-to-one.

(b) A can be viewed as a mapping from R^4 to R^3, thus multiplication by A is not one-to-one.

(c) A reduces to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, so nullity$(A) = 0$ or $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, so multiplication by A is one-to-one.

5. (a) All points on the line $y = -x$ are mapped to 0, so $\ker(T) = \{k(-1, 1)\}$.

(b) Since $\ker(T) \neq \{\mathbf{0}\}$, T is not one-to-one.

7. (a) Since nullity$(T) = 0$, T is one-to-one.

(b) nullity$(T) = n - (n - 1) = 1$, so T is not one-to-one.

(c) Since $n < m$, T is not one-to-one.

(d) Since $R(T) = R^n$, rank$(T) = n$, so nullity $(T) = 0$. Thus T is one-to-one.

9. If there were such a transformation T, then it would have nullity 0 (because it would be one-to-one), and have rank no greater than the dimension of W (because its range is a subspace W). Then by the dimension Theorem, $\dim(V) = \text{rank}(T) + \text{nullity}(T) \leq \dim(W)$ which contradicts that $\dim(W) < \dim(V)$. Thus there is no such transformation.

11. (a) $T\left(\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}$

(b) $T_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and

$T_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix}.$

(c) If $p(x)$ is a polynomial and $p(0) = 0$, then $p(x)$ has constant term 0.

$T(ax^3 + bx^2 + cx) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

(d) $T(a+b\sin(x)+c\cos(x))=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

13. $T(k\mathbf{f})=kf(0)+2kf'(0)+3kf'(1)=kT(\mathbf{f})$
$T(\mathbf{f}+\mathbf{g})=f(0)+g(0)+2(f'(0)+g'(0))$
$\qquad +3(f'(1)+g'(1))$
$\qquad =T(\mathbf{f})+T(\mathbf{g})$
Thus T is a linear transformation.
Let $\mathbf{f}=f(x)=x^2(x-1)^2$, then
$f'(x)=2x(x-1)(2x-1)$ so $f(0)=0$, $f'(0)=0$, and $f'(1)=0$.
$T(\mathbf{f})=f(0)+2f'(0)+3f'(1)=0$, so
$\ker(T)\neq\{\mathbf{0}\}$ and T is not one-to-one.

15. Yes; if $T(a, b, c) = \mathbf{0}$, then $a = b = c = 0$, so $\ker(T) = \{\mathbf{0}\}$.

17. No; T is not one-to-one because $\ker(T) \neq \{\mathbf{0}\}$ as $T(\mathbf{a}) = \mathbf{a} \times \mathbf{a} = \mathbf{0}$.

19. Yes, since $\langle T(\mathbf{u}), T(\mathbf{v})\rangle=\langle \mathbf{u}, \mathbf{v}\rangle$, so
$\|T(\mathbf{u})\|=\sqrt{\langle T(\mathbf{u}), T(\mathbf{u})\rangle}=\sqrt{\langle \mathbf{u}, \mathbf{u}\rangle}=\|\mathbf{u}\|=1$ and
$T(\mathbf{u})$ is orthogonal to $T(\mathbf{v})$ if $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

True/False 8.2

(a) False; $\dim(R^2)=2$ while $\dim(P_2)=3$.

(b) True; if $\ker(T) = \{\mathbf{0}\}$ is one-to-one $\operatorname{rank}(T) = 4$ so T is one-to-one and onto.

(c) False; $\dim(M_{33})=9$ while $\dim(P_9)=10$.

(d) True; for instance, if V consists of all matrices of the form $\begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix}$, then

$T\left(\begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix}\right)=\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is an isomorphism T:

$V\to R^4$.

(e) False; $T(I) = P - P = \mathbf{0}$ so $\ker(T) \neq \mathbf{0}$.

(f) False; if there were such a transformation T, then $\dim(\ker(T)) = \dim(R(T))$, but since $\dim(\ker(T))+\dim(R(T))=\dim(P_4)=5$, there can be no such transformation.

Section 8.3

Exercise Set 8.3

1. (a) $(T_2\circ T_1)(x, y)=T_2(2x, 3y)$
$\qquad =(2x-3y, 2x+3y)$

(b) $(T_2\circ T_1)(x, y)$
$=T_2(x-3y, 0)$
$=(4(x-3y)-5(0), 3(x-3y)-6(0))$
$=(4x-12y, 3x-9y)$

(c) $(T_2\circ T_1)(x, y)=T_2(2x, -3y, x+y)$
$\qquad =(2x+3y, -3y+x+y)$
$\qquad =(2x+3y, x-2y)$

(d) $(T_2\circ T_1)(x, y)$
$=T_2(x-y, y, x)$
$=(0, (x-y)+y+x)$
$=(0, 2x)$

3. (a) $(T_1\circ T_2)(A)=T_1(A^T)=\operatorname{tr}\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right)=a+d$

(b) $(T_2\circ T_1)(A)$ does not exist because $T_1(A)$ is not a 2×2 matrix.

5. $T_2(\mathbf{v})=\frac{1}{4}\mathbf{v}$, then

$(T_1\circ T_2)(\mathbf{v})=T_1\left(\frac{1}{4}\mathbf{v}\right)=4\left(\frac{1}{4}\mathbf{v}\right)=\mathbf{v}$ and

$(T_2\circ T_1)(\mathbf{v})=T_2(4\mathbf{v})=\frac{1}{4}(4\mathbf{v})=\mathbf{v}$.

9. By inspection, $T(x, y, z) = (x, y, 0)$.
Then
$T(T(x, y, z)) = T(x, y, 0) = (x, y, 0) = T(x, y, z)$ or
$T\circ T=T$.

11. (a) Since $\det(A) = 0$, T does not have an inverse.

(b) $\det(A) = -8 \neq 0$, so T has an inverse.

$$A^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & -\frac{3}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} \\ -\frac{3}{8} & \frac{5}{8} & \frac{1}{4} \end{bmatrix} \text{ so}$$

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{8}x_1 + \frac{1}{8}x_2 - \frac{3}{4}x_3 \\ \frac{1}{8}x_1 + \frac{1}{8}x_2 + \frac{1}{4}x_3 \\ -\frac{3}{8}x_1 + \frac{5}{8}x_2 + \frac{1}{4}x_3 \end{bmatrix}.$$

(c) $\det(A) = -2 \neq 0$, so T has an inverse.

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \text{ so}$$

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ -\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \end{bmatrix}.$$

(d) $\det(A) = 1 \neq 0$, so T has an inverse.

$$A^{-1} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} \text{ so}$$

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 + 3x_2 - x_3 \\ -2x_1 - 2x_2 + x_3 \\ -4x_1 - 5x_2 + 2x_3 \end{bmatrix}.$$

13. (a) For T to have an inverse, all the a_i, $i = 1, 2, ..., n$ must be nonzero.

(b) $T^{-1}(x_1, x_2, ..., x_n)$

$$= \left(\frac{1}{a_1}x_1, \frac{1}{a_2}x_2, ..., \frac{1}{a_n}x_n\right)$$

15. (a) Since $T_1(p(x)) = xp(x)$,

$$T_1^{-1}(p(x)) = \frac{1}{x}p(x).$$

Since $T_2(p(x)) = p(x+1)$,

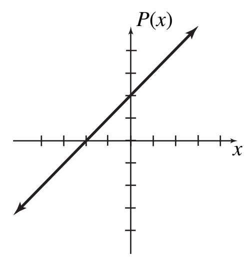
$$T_2^{-1}(p(x)) = p(x-1).$$

$$\begin{aligned}(T_2 \circ T_1)^{-1}(p(x)) &= (T_1^{-1} \circ T_2^{-1})(p(x)) \\ &= T_1^{-1}(p(x-1)) \\ &= \frac{1}{x}p(x-1)\end{aligned}$$

17. (a) $T(1 - 2x) = (1 - 2(0), 1 - 2(1)) = (1, -1)$

(c) Let $p(x) = a_0 + a_1x$, then $T(p(x)) = (a_0, a_0 + a_1)$ so if $T(p(x)) = (0, 0)$, then $a_0 = a_1 = 0$ and p is the zero polynomial, so $\ker(T) = \{\mathbf{0}\}$.

(d) Since $T(p(x)) = (a_0, a_0 + a_1)$, then $T^{-1}(2, 3)$ has $a_0 = 2$ and $a_0 + a_1 = 3$ or $a_1 = 1$. Thus, $T^{-1}(2, 3) = 2 + x$.



21. (a) $T_1(x, y) = (x, -y)$ and $T_2(x, y) = (-x, y)$

$(T_1 \circ T_2)(x, y) = T_1(-x, y) = (-x, -y)$

$(T_2 \circ T_1)(x, y) = T_2(x, -y) = (-x, -y)$

$T_1 \circ T_2 = T_2 \circ T_1$

(b) Consider $(x, y) = (0, 1)$:

$(T_2 \circ T_1)(0, 1) = T_2(0, 0) = (0, 0)$ but

$$\begin{aligned}(T_1 \circ T_2)(0, 1) &= T_1(-\sin\theta, \cos\theta) \\ &= (-\sin\theta, 0)\end{aligned}$$

Thus $T_1 \circ T_2 \neq T_2 \circ T_1$.

(c) $T_1(x, y, z) = (kx, ky, kz)$ and $T_2(x, y, z) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z)$
$(T_1 \circ T_2)(x, y, z) = (k(x\cos\theta - y\sin\theta), k(x\sin\theta + y\sin\theta), kz)$
$(T_2 \circ T_1)(x, y, z) = (kx\cos\theta - ky\sin\theta, kx\sin\theta + ky\cos\theta, kz)$
$T_1 \circ T_2 = T_2 \circ T$

True/False 8.3

(a) True

(b) False; the linear operators in Exercise 21(b) are an example.

(c) False; a linear transformation need not have an inverse.

(d) True; for T to have an inverse, it must be one-to-one.

(e) False; T^{-1} does not exist.

(f) True; if T_1 is not one-to-one then there is some nonzero vector $\mathbf{v}_1$ with $T_1(\mathbf{v}_1) = \mathbf{0}$, then $(T_2 \circ T_1)(\mathbf{v}_1) = T_2(\mathbf{0}) = \mathbf{0}$ and $\ker(T_2 \circ T_1) \neq \{\mathbf{0}\}$.

Section 8.4

Exercise Set 8.4

1. (a) $T(\mathbf{u}_1) = T(1) = x = \mathbf{v}_2$
$T(\mathbf{u}_2) = T(x) = x^2 = \mathbf{v}_3$
$T(\mathbf{u}_3) = T(x^2) = x^3 = \mathbf{v}_4$

Thus the matrix for T relative to B and B' is $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $[T]_{B', B}[\mathbf{x}]_B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ c_0 \\ c_1 \\ c_2 \end{bmatrix}$

$[T(\mathbf{x})]_{B'} = [c_0x + c_1x^2 + c_2x^3]_{B'} = \begin{bmatrix} 0 \\ c_0 \\ c_1 \\ c_2 \end{bmatrix}$

3. (a) $T(1) = 1$
$T(x) = x - 1 = -1 + x$
$T(x^2) = (x-1)^2 = 1 - 2x + x^2$

Thus the matrix for T relative to B is $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $[T]_B[\mathbf{x}]_B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_0 - a_1 + a_2 \\ a_1 - 2a_2 \\ a_2 \end{bmatrix}$

For $\mathbf{x} = a_0 + a_1x + a_2x^2$,

$$T(\mathbf{x}) = a_0 + a_1(x-1) + a_2(x-1)^2 = a_0 - a_1 + a_2 + (a_1 - 2a_2)x + a_2x^2,$$

so $[T(\mathbf{x})]_B = \begin{bmatrix} a_0 - a_1 + a_2 \\ a_1 - 2a_2 \\ a_2 \end{bmatrix}$.

5. **(a)** $T(\mathbf{u}_1) = T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} = 0\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 + \frac{8}{3}\mathbf{v}_2$

$T(\mathbf{u}_2) = T\left(\begin{bmatrix} -2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} = 0\mathbf{v}_1 + \mathbf{v}_2 + \frac{4}{3}\mathbf{v}_3$

$[T]_{B',B} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & 1 \\ \frac{8}{3} & \frac{4}{3} \end{bmatrix}$

7. **(a)** $T(1) = 1$

$T(x) = 2x + 1 = 1 + 2x$

$T(x)^2 = (2x+1)^2 = 1 + 4x + 4x^2$

$[T]_B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

(b) The matrix for $2 - 3x + 4x^2$ is $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$.

$[T]_B\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 16 \end{bmatrix}$

Thus $T(2 - 3x + 4x^2) = 3 + 10x + 16x^2$.

(c) $T(2 - 3x + 4x^2) = 2 - 3(2x+1) + 4(2x+1)^2 = 2 - 6x - 3 + 16x^2 + 16x + 4 = 3 + 10x + 16x^2$

9. **(a)** Since A is the matrix for T relative to B, $A = [[T(\mathbf{v}_1)]_B \mid [T(\mathbf{v}_2)]_B]$. That is, $[T(\mathbf{v}_1)]_B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $[T(\mathbf{v}_2)]_B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

(b) Since $[T(\mathbf{v}_1)]_B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$,

$T(\mathbf{v}_1) = 1\mathbf{v}_1 - 2\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

Similarly,

$T(\mathbf{v}_2) = 3\mathbf{v}_1 + 5\mathbf{v}_2 = \begin{bmatrix} 3 \\ 9 \end{bmatrix} + \begin{bmatrix} -5 \\ 20 \end{bmatrix} = \begin{bmatrix} -2 \\ 29 \end{bmatrix}$.

(c) If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 4 \end{bmatrix}$,

then $\begin{aligned} x_1 &= c_1 - c_2 \\ x_2 &= 3c_1 + 4c_2 \end{aligned}$. Solving for c_1 and c_2 gives $c_1 = \frac{4}{7}x_1 + \frac{1}{7}x_2$, $c_2 = -\frac{3}{7}x_1 + \frac{1}{7}x_2$, so

$$\begin{aligned} T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) \\ &= \left(\frac{4}{7}x_1 + \frac{1}{7}x_2\right)\begin{bmatrix} 3 \\ -5 \end{bmatrix} + \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right)\begin{bmatrix} -2 \\ 29 \end{bmatrix} \\ &= \begin{bmatrix} \frac{18}{7}x_1 + \frac{1}{7}x_2 \\ -\frac{107}{7}x_1 + \frac{24}{7}x_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{18}{7} & \frac{1}{7} \\ -\frac{107}{7} & \frac{24}{7} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

(d) $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{18}{7} & \frac{1}{7} \\ -\frac{107}{7} & \frac{24}{7} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{19}{7} \\ -\frac{83}{7} \end{bmatrix}$

11. **(a)** Since A is the matrix for T relative to B, the columns of A are $[T(\mathbf{v}_1)]_B$, $[T(\mathbf{v}_2)]_B$, and $[T(\mathbf{v}_3)]_B$, respectively. That is,

$[T(\mathbf{v}_1)]_B = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$, $[T(\mathbf{v}_2)]_B = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$, and

$[T(\mathbf{v}_3)]_B = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$.

(b) Since $[T(\mathbf{v}_1)]_B = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$,

$$\begin{aligned} &T(\mathbf{v}_1) \\ &= \mathbf{v}_1 + 2\mathbf{v}_2 + 6\mathbf{v}_3 \\ &= 3x + 3x^2 - 2 + 6x + 4x^2 + 18 + 42x + 12x^2 \\ &= 16 + 51x + 19x^2. \end{aligned}$$

Similarly, $T(\mathbf{v}_2) = 3\mathbf{v}_1 - 2\mathbf{v}_3$
$$= 9x + 9x^2 - 6 - 14x - 4x^2$$
$$= -6 - 5x + 5x^2,$$
and $T(\mathbf{v}_3) = -\mathbf{v}_1 + 5\mathbf{v}_2 + 4\mathbf{v}_3$
$$\begin{aligned} &= -3x - 3x^2 - 5 + 15x + 10x^2 + 12 \\ &\quad + 28x + 8x^2 \\ &= 7 + 40x + 15x^2. \end{aligned}$$

(c) If $a_0 + a_1x + a_2x^2$
$$\begin{aligned} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \\ &= c_1(3x + 3x^2) + c_2(-1 + 3x + 2x^2) \\ &\quad + c_3(3 + 7x + 2x^2), \end{aligned}$$

Then $\begin{aligned} a_0 &= -c_2 + 3c_3 \\ a_1 &= 3c_1 + 3c_2 + 7c_3 \\ a_2 &= 3c_1 + 2c_2 + 2c_3 \end{aligned}$.

Solving for c_1, c_2, and c_3 gives

$c_1 = \frac{1}{3}(a_0 - a_1 + 2a_2)$,

$c_2 = \frac{1}{8}(-5a_0 + 3a_1 - 3a_2)$,

$c_3 = \frac{1}{8}(a_0 + a_1 - a_2)$, so

$$\begin{aligned} &T(a_0 + a,\ x + a_2x^2) \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) \\ &= \frac{1}{3}(a_0 - a_1 + 2a_2)(16 + 51x + 19x^2) \\ &\quad + \frac{1}{8}(-5a_0 + 3a_1 - 3a_2)(-6 - 5x + 5x^2) \\ &\quad + \frac{1}{8}(a_0 + a_1 - a_2)(7 + 40x + 15x^2) \\ &= \frac{239a_0 - 161a_1 + 289a_2}{24} \\ &\quad + \frac{201a_0 - 111a_1 + 247a_2}{8}x \\ &\quad + \frac{61a_0 - 31a_1 + 107a_2}{12}x^2 \end{aligned}$$

(d) In $1 + x^2$, $a_0 = 1$, $a_1 = 0$, $a_2 = 1$.
$$\begin{aligned} &T(1 + x^2) \\ &= \frac{239 + 289}{24} + \frac{201 + 247}{8}x + \frac{61 + 107}{12}x^2 \\ &= 22 + 56x + 14x^2 \end{aligned}$$

13. **(a)** $(T_2 \circ T_1)(1) = T_2(2) = 6x$

$(T_2 \circ T_1)(x) = T_2(-3x) = -9x^2$

Thus, $[T_2 \circ T_1]_{B',\,B} = \begin{bmatrix} 0 & 0 \\ 6 & 0 \\ 0 & -9 \\ 0 & 0 \end{bmatrix}$.

$T_1(1) = 2$

$T_1(x) = -3x$

Thus $[T_1]_{B'',\,B} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$.

$T_2(1) = 3x$

$T_2(x) = 3x^2$

$T_2(x^2) = 3x^3$

Thus $[T_2]_{B',\,B''} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(b) $[T_2 \circ T_1]_{B',\,B} = [T_2]_{B',\,B''}[T_1]_{B'',\,B}$

(c) $\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 6 & 0 \\ 0 & -9 \\ 0 & 0 \end{bmatrix}$

15. If T is a contraction or dilation of V, then T maps any basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of V to $\{k\mathbf{v}_1, k\mathbf{v}_2, \ldots, k\mathbf{v}_n\}$ where k is a nonzero constant. Thus the matrix for T relative to B is
$$\begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & k \end{bmatrix}.$$

17. The matrix for T relative to B is the matrix whose columns are the transforms of the basis vectors in B in terms of the standard basis. Since B is the standard basis for R^n, this matrix is the standard matrix for T. Also, since B' is the standard basis for R^m, the resulting transformation will give vector components relative to the standard basis.

19. **(a)** $D(\mathbf{f}_1) = D(1) = 0$
$D(\mathbf{f}_2) = D(\sin x) = \cos x$
$D(\mathbf{f}_3) = D(\cos x) = -\sin x$
The matrix for D relative to this basis is
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$

(b) $D(\mathbf{f}_1) = D(1) = 0$
$D(\mathbf{f}_2) = D(e^x) = e^x$
$D(\mathbf{f}_3) = D(e^{2x}) = 2e^{2x}$
The matrix for D relative to this basis is
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$

(c) $D(\mathbf{f}_1) = D(e^{2x}) = 2e^{2x}$
$D(\mathbf{f}_2) = D(xe^{2x}) = e^{2x} + 2xe^{2x}$
$D(\mathbf{f}_3) = D(x^2e^{2x}) = 2xe^{2x} + 2x^2e^{2x}$
The matrix for D relative to this basis is
$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$

(d) $D(4e^{2x} + 6xe^{2x} - 10x^2e^{2x})$
$$= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ -10 \end{bmatrix} = \begin{bmatrix} 14 \\ -8 \\ -20 \end{bmatrix}$$
Thus, $D(4e^{2x} + 6xe^{2x} - 10x^2e^{2x}) = 14e^{2x} - 8xe^{2x} - 20x^2e^{2x}$.

21. **(a)** $[T_2 \circ T_1]_{B', B} = [T_2]_{B', B''}[T_1]_{B'', B}$

(b) $[T_3 \circ T_2 \circ T_1]_{B', B}$
$= [T_3]_{B', B'''}[T_2]_{B''', B''}[T_1]_{B'', B}$

True/False 8.4

(a) False; the conclusion would only be true if $T: V \to V$ were a linear operator, i.e., if $V = W$.

(b) False; the conclusion would only be true if $T: V \to V$ were a linear operator, i.e., if $V = W$.

(c) True; since the matrix for T is invertible, $\ker(T) = \{\mathbf{0}\}$.

(d) False; the matrix of $S \circ T$ relative to B is $[S]_B[T]_B$.

(e) True

Section 8.5

Exercise Set 8.5

1. $T(\mathbf{u}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T(\mathbf{u}_2) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, so
$[T]_B = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}.$
By inspection, $\mathbf{v}_1 = 2\mathbf{u}_1 + \mathbf{u}_2$ and $\mathbf{v}_2 = -3\mathbf{u}_1 + 4\mathbf{u}_2$, so the transition matrix from B' to B is $P = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$. Thus
$P_{B \to B'} = P^{-1} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix}.$
$$[T]_{B'} = P_{B \to B'}[T]_B P_{B' \to B} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{11} & -\frac{56}{11} \\ -\frac{2}{11} & \frac{3}{11} \end{bmatrix}$$

3. Since B is the standard basis for R^2,

$$[T]_B = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$ The matrices for $P_{B \to B'}$ and $P_{B' \leftrightarrow B}$ are the same as in Exercise 1, so

$$\begin{aligned}[T]_{B'} &= P_{B \to B'}[T]_B P_{B' \to B} \\ &= \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{13}{11\sqrt{2}} & -\frac{25}{11\sqrt{2}} \\ \frac{5}{11\sqrt{2}} & \frac{9}{11\sqrt{2}} \end{bmatrix}.\end{aligned}$$

5. $T(\mathbf{u}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $T(\mathbf{u}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $T(\mathbf{u}_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so

$$[T]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By inspection, $\mathbf{v}_1 = \mathbf{u}_1$, $\mathbf{v}_2 = \mathbf{u}_1 + \mathbf{u}_2$, and $\mathbf{v}_3 = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$, so the transition matrix from B' to B is $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$

Thus $P_{B \to B'} = P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and

$$\begin{aligned}[T]_{B'} &= P_{B \to B'}[T]_B P_{B' \to B} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.\end{aligned}$$

7. $T(\mathbf{p}_1) = 6 + 3(x+1) = 9 + 3x = \frac{2}{3}\mathbf{p}_1 + \frac{1}{2}\mathbf{p}_2$, and

$T(\mathbf{p}_2) = 10 + 2(x+1) = 12 + 2x = -\frac{2}{9}\mathbf{p}_1 + \frac{4}{3}\mathbf{p}_2$,

so $[T]_B = \begin{bmatrix} \frac{2}{3} & -\frac{2}{9} \\ \frac{1}{2} & \frac{4}{3} \end{bmatrix}.$

$\mathbf{q}_1 = -\frac{2}{9}\mathbf{p}_1 + \frac{1}{3}\mathbf{p}_2$ and $\mathbf{q}_2 = \frac{7}{9}\mathbf{p}_1 - \frac{1}{6}\mathbf{p}_2$, so the transition matrix from B' to B is

$P = \begin{bmatrix} -\frac{2}{9} & \frac{7}{9} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$. Thus $P_{B \to B'} = P^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$

and
$$\begin{aligned}[T]_{B'} &= P_{B \to B'}[T]_B P_{B' \to B} \\ &= \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{2}{9} \\ \frac{1}{2} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} -\frac{2}{9} & \frac{7}{9} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.\end{aligned}$$

11. (a) The matrix for T relative to the standard basis B is $[T]_B = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. The eigenvalues of $[T]_B$ are $\lambda = 2$ and $\lambda = 3$ with corresponding eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Then for $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, we have

$P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ and $P^{-1}[T]_B P = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Since P represents the transition matrix from the basis B' to the standard basis B, then $B' = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ is a basis for which $[T]_{B'}$ is diagonal.

(b) The matrix for T relative to the standard basis B is $[T]_B = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$.

The eigenvalues of $[T]_B$ are $\lambda = \frac{5+\sqrt{21}}{2}$ and $\lambda = \frac{5-\sqrt{21}}{2}$ with corresponding eigenvectors $\begin{bmatrix} \frac{-3-\sqrt{21}}{6} \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{-3+\sqrt{21}}{6} \\ 1 \end{bmatrix}$.

Then for $P = \begin{bmatrix} \frac{-3-\sqrt{21}}{6} & \frac{-3+\sqrt{21}}{6} \\ 1 & 1 \end{bmatrix}$, we have

$P^{-1} = \begin{bmatrix} -\frac{3}{\sqrt{21}} & \frac{-3+\sqrt{21}}{2\sqrt{21}} \\ \frac{3}{\sqrt{21}} & \frac{3+\sqrt{21}}{2\sqrt{21}} \end{bmatrix}$ and

$P^{-1}[I]_B P = \begin{bmatrix} \frac{5+\sqrt{21}}{2} & 0 \\ 0 & \frac{5-\sqrt{21}}{2} \end{bmatrix}$. Since P represents the transition matrix from the basis B' to the standard basis B, then $B' = \left\{ \begin{bmatrix} \frac{-3-\sqrt{21}}{6} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{-3+\sqrt{21}}{6} \\ 1 \end{bmatrix} \right\}$ is a basis for which $[T]_{B'}$ is diagonal.

13. **(a)** $T(1) = 5 + x^2$
$T(x) = 6 - x$
$T(x^2) = 2 - 8x - 2x^2$
Thus the matrix for T relative to the standard basis B is $[T]_B = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$. The characteristic equation for $[T]_B$ is $\lambda^3 - 2\lambda^2 - 15\lambda + 36 = (\lambda+4)(\lambda-3)^2 = 0$, so the eigenvalues of T are $\lambda = -4$ and $\lambda = 3$.

(b) A basis for the eigenspace corresponding to $\lambda = -4$ is $\begin{bmatrix} -2 \\ \frac{8}{3} \\ 1 \end{bmatrix}$, so the polynomial $-2 + \frac{8}{3}x + x^2$ is a basis in P^2. A basis for the eigenspace corresponding to $\lambda = 3$ is $\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, so the polynomial so the polynomial $5 - 2x + x^2$ is a basis in P^2.

15. If $\mathbf{v}$ is an eigenvector of T corresponding to λ, then $\mathbf{v}$ is a nonzero vector which satisfies the equation $T(\mathbf{v}) = \lambda\mathbf{v}$ or $(\lambda I - T)(\mathbf{v}) = \mathbf{0}$. Thus $\mathbf{v}$ is in the kernel of $\lambda I - T$.

17. Since $C[\mathbf{x}]_B = D[\mathbf{x}]_B$ for all $\mathbf{x}$ in V, then we can let $\mathbf{x} = \mathbf{v}_i$ for each of the basis vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ of V. Since $[\mathbf{v}_i]_B = \mathbf{e}_i$ for each i where $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ is the standard basis for R^n, this yields $C\mathbf{e}_i = D\mathbf{e}_i$ for $i = 1, 2, ..., n$. But $C\mathbf{e}_i$ and $D\mathbf{e}_i$ are the ith columns of C and D, respectively. Since corresponding columns of C and D are all equal, $C = D$.

True/False 8.5

(a) False; every matrix is similar to itself since $A = I^{-1}AI$.

(b) True; if $A = P^{-1}BP$ and $B = Q^{-1}CQ$, then $A = P^{-1}(Q^{-1}CQ)P = (QP)^{-1}C(QP)$.

(c) True; invertibility is a similarity invariant.

(d) True; if $A = P^{-1}BP$, then $A^{-1} = (P^{-1}BP)^{-1} = P^{-1}B^{-1}P$.

(e) True

(f) False; for example, let T be the zero operator.

(g) True

(h) False; if B and B' are different, let $[T]_B$ be given by the matrix $P_{B\to B'}$. Then $[T]_{B', B} = P_{B\to B'}[T]_B = P_{B\to B'}P_{B'\to B} = I_n$.

Chapter 8 Supplementary Exercises

1. No; $$\begin{aligned} T(\mathbf{x}_1 + \mathbf{x}_2) &= A(\mathbf{x}_1 + \mathbf{x}_2) + B \\ &\neq (A\mathbf{x}_1 + B) + (A\mathbf{x}_2 + B) \\ &= T(\mathbf{x}_1) + T(\mathbf{x}_2), \end{aligned}$$
and if $c \neq 1$, then
$T(c\mathbf{x}) = cA\mathbf{x} + B \neq c(A\mathbf{x} + B) = cT(\mathbf{x})$.

3. $T(k\mathbf{v}) = \|k\mathbf{v}\|k\mathbf{v} = k|k|\|\mathbf{v}\|\mathbf{v} \neq kT(\mathbf{v})$ if $k \neq 1$.

5. **(a)** The matrix for T relative to the standard basis is $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$. A basis for the range of T is a basis for the column space of A. A reduces to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.
Since row operations don't change the dependency relations among columns, the reduced form of A indicates that $T(\mathbf{e}_3)$ and any two of $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, $T(\mathbf{e}_4)$ form a basis for the range.
The reduced form of A shows that the general solution of $A\mathbf{x} = \mathbf{0}$ is $x = -s$, $x_2 = s$,

$x_3 = 0,\ \ x_4 = s$ so a basis for the null space of *A*, which is the kernel of *T* is $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

(b) Since *R*(*T*) is three-dimensional and ker(*T*) is one-dimensional, rank(*T*) = 3 and nullity(*T*) = 1.

7. (a) The matrix for *T* relative to *B* is

$$[T]_B = \begin{bmatrix} 1 & 1 & 2 & -2 \\ 1 & -1 & -4 & 6 \\ 1 & 2 & 5 & -6 \\ 3 & 2 & 3 & -2 \end{bmatrix}.$$

$[T]_B$ reduces to $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which has rank 2 and nullity 2. Thus, rank(*T*) = 2 and nullity(*T*) = 2.

(b) Since rank(*T*) < dim(*V*), *T* is not one-to-one.

9. (a) Since $A = P^{-1}BP,$ then

$$\begin{aligned} A^T &= (P^{-1}BP)^T = P^T B^T (P^{-1})^T \\ &= ((P^T)^{-1})^{-1} B^T (P^{-1})^T \\ &= ((P^{-1})^T)^{-1} B^T (P^{-1})^T. \end{aligned}$$

Thus, A^T and B^T are similar.

(b) $A^{-1} = (P^{-1}BP)^{-1} = P^{-1}B^{-1}P$

Thus A^{-1} and B^{-1} are similar.

11. For $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$

$$\begin{aligned} T(X) &= \begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} b & b \\ d & d \end{bmatrix} \\ &= \begin{bmatrix} a+b+c & 2b+d \\ d & d \end{bmatrix}. \end{aligned}$$

$T(X) = \mathbf{0}$ gives the equations

$$\begin{aligned} a+b+c &= 0 \\ 2b+d &= 0 \\ d &= 0. \end{aligned}$$

Thus $c = -a$ and *X* is in ker(*T*) if it has the form $\begin{bmatrix} k & 0 \\ -k & 0 \end{bmatrix},$ so $\ker(T) = \left\{ k \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \right\}$ which is one-dimensional. Hence, nullity(*T*) = 1 and $\text{rank}(T) = \dim(M_{22}) - \text{nullity}(T) = 3.$

13. The standard basis for M_{22} is $\mathbf{u}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$ $\mathbf{u}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$ $\mathbf{u}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$ $\mathbf{u}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$ Since $L(\mathbf{u}_1) = \mathbf{u}_1,$ $L(\mathbf{u}_2) = \mathbf{u}_3,$ $L(\mathbf{u}_3) = \mathbf{u}_2,$ and $L(\mathbf{u}_4) = \mathbf{u}_4,$ the matrix of *L* relative to the standard basis is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

15. The transition matrix from B' to *B* is $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$ thus

$$[T]_{B'} = P^{-1}[T]_B P = \begin{bmatrix} -4 & 0 & 9 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}.$$

17. $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$ $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$ thus $[T\}_B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$

Note that this can also be read directly from $[T(\mathbf{x})]_B$.

19. (a) $D(\mathbf{f}+\mathbf{g}) = (f(x)+g(x))'' = f''(x)+g''(x)$ and $D(k\mathbf{f}) = (kf(x))'' = kf''(x).$

(b) If **f** is in ker(*D*), then **f** has the form $\mathbf{f} = f(x) = a_0 + a_1 x,$ so a basis for ker(*D*) is *f*(*x*) = 1, *g*(*x*) = *x*.

(c) *D*(**f**) = **f** if and only if $\mathbf{f} = f(x) = ae^x + be^{-x}$ for arbitrary constants *a* and *b*. Thus, $f(x) = e^x$ and $g(x) = e^{-x}$ form a basis for this subspace.

21. **(c)** Note that $a_1P_1(x)+a_2P_2(x)+a_3P_3(x)$ evaluated at x_1, x_2, and x_3 gives the values a_1, a_2, and a_3, respectively, since $P_i(x_i)=1$ and $P_i(x_j)=0$ for $i \neq j$.

Thus

$$T(a_1P_1(x)+a_2P_2(x)+a_3P_3(x))=\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \text{ or}$$

$$T^{-1}\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right)=a_1P_1(x)+a_2P_2(x)+a_3P_3(x).$$

(d) From the computations in part (c), the points lie on the graph.

23.
$$\begin{aligned} D(1) &= 0 \\ D(x) &= 1 \\ D(x^2) &= 2x \\ &\vdots \\ D(x^n) &= nx^{n-1} \end{aligned}$$

This gives the matrix shown.

25.
$$\begin{aligned} J(1) &= x \\ J(x) &= \frac{1}{2}x^2 \\ J(x^2) &= \frac{1}{3}x^3 \\ &\vdots \\ J(x^n) &= \frac{1}{n+1}x^{n+1} \end{aligned}$$

This gives the $(n + 2) \times (n + 1)$ matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n+1} \end{bmatrix}.$$

Chapter 9

Numerical Methods

Section 9.1

Exercise Set 9.1

1. In matrix form, the system is
$$\begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
which reduces to $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and
$\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The second system is $\begin{aligned} 3y_1 &= 0 \\ -2y_1 + y_2 &= 1 \end{aligned}$, which has solution $y_1 = 0$, $y_2 = 1$, so the first system becomes $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\begin{aligned} x_1 - 2x_2 &= 0 \\ x_2 &= 1 \end{aligned}$, which gives the solution to the original system: $x_1 = 2$, $x_2 = 1$.

3. Reduce the coefficient matrix.
$$\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$
$= \mathbf{U}$

The multipliers used were $\frac{1}{2}$, 1, and $\frac{1}{3}$, which leads to $L = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$, so the system is
$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$
$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ is $\begin{aligned} 2y_1 &= -2 \\ -y_1 + 3y_2 &= -2 \end{aligned}$ which has the solution $y_1 = -1$, $y_2 = -1$.

$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is $\begin{aligned} x_1 + 4x_2 &= -1 \\ x_2 &= -1 \end{aligned}$ which gives the solution to the original system: $x_1 = 3$, $x_2 = -1$.

5. Reduce the coefficient matrix.
$$\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 4 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$
The multipliers used were $\frac{1}{2}$, 1, $-\frac{1}{2}$, -4, and $\frac{1}{5}$, which leads to $L = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}$, so the system is
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}.$$
$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$ is
$$\begin{aligned} 2y_1 \qquad\qquad &= -4 \\ -2y_2 \qquad &= -2 \\ -y_1 + 4y_2 + 5y_3 &= 6 \end{aligned}$$
which has the solution $y_1 = -2$, $y_2 = 1$, $y_3 = 0$.

$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is $\begin{aligned} x_1 - x_2 - x_3 &= -2 \\ x_2 - x_3 &= 1 \\ x_3 &= 0 \end{aligned}$

which gives the solution to the original system: $x_1 = -1$, $x_2 = 1$, $x_3 = 0$.

7. Reduce the coefficient matrix.
$$\begin{bmatrix} 5 & 5 & 10 \\ -8 & -7 & -9 \\ 0 & 4 & 26 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ -8 & -7 & -9 \\ 0 & 4 & 26 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 7 \\ 0 & 4 & 26 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} = U$$
The multipliers used were $\frac{1}{5}$, 8, -4, and $-\frac{1}{2}$

which leads to $L = \begin{bmatrix} 5 & 0 & 0 \\ -8 & 1 & 0 \\ 0 & 4 & -2 \end{bmatrix}$, so the system is $\begin{bmatrix} 5 & 0 & 0 \\ -8 & 1 & 0 \\ 0 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$.

$\begin{bmatrix} 5 & 0 & 0 \\ -8 & 1 & 0 \\ 0 & 4 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ is

$\begin{aligned} 5y_1 \quad\quad &= 0 \\ -8y_1 + y_2 \quad &= 1 \\ 4y_2 - 2y_3 &= 4 \end{aligned}$ which has the solution

$y_1 = 0,\ \ y_2 = 1,\ \ y_3 = 0.$

$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is $\begin{aligned} x_1 + x_2 + 2x_3 &= 0 \\ x_2 + 7x_3 &= 1 \\ x_3 &= 0 \end{aligned}$

which gives the solution to the original system: $x_1 = -1,\ \ x_2 = 1,\ \ x_3 = 0.$

9. Reduce the coefficient matrix.

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

The multipliers used were -1, -2, $\frac{1}{3}$, 1, $\frac{1}{2}$, -1, and $\frac{1}{4}$, which leads to $L = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$, so the system is

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix}.$$

$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix}$ is

$\begin{aligned} -y_1 \quad\quad\quad &= 5 \\ 2y_1 + 3y_2 \quad\quad &= -1 \\ -y_2 + 2y_3 \quad &= 3 \\ y_3 + 4y_4 &= 7 \end{aligned}$, which has the solution $y_1 = -5,\ \ y_2 = 3,\ \ y_3 = 3,\ \ y_4 = 1.$

$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 3 \\ 1 \end{bmatrix}$ is

$\begin{aligned} x_1 - x_3 \quad &= -5 \\ x_2 + 2x_4 &= 3 \\ x_3 + x_4 &= 3 \\ x_4 &= 1 \end{aligned}$, which gives the solution to the original system: $x_1 = -3,\ \ x_2 = 1,\ \ x_3 = 2,\ \ x_4 = 1.$

11. (a) Reduce A to upper triangular form.

$$\begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{2}$, 2, and -2, which leads to $L = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ where the 1's on the diagonal reflect that no multiplication was required on the 2nd and 3rd diagonal entries.

(b) To change the 2 on the diagonal of L to a 1, the first column of L is divided by 2 and the diagonal matrix has a 2 as the 1, 1 entry.

$$A = L_1DU_1$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

is the LDU-decomposition of A.

(c) Let $U_2 = DU$, and $L_2 = L_1$, then

$$A = L_2U_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

13. Reduce A to upper triangular form.

$$\begin{bmatrix} 3 & -12 & 6 \\ 0 & 2 & 0 \\ 6 & -28 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 6 & -28 & 13 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{3}$, -6, $\frac{1}{2}$, and 4,

which leads to $L_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 6 & -4 & 1 \end{bmatrix}$. Since

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is the}$$

LDU-decomposition of A.

15. $P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, so the

system $P^{-1}A\mathbf{x} = P^{-1}\mathbf{b}$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \text{ is } \begin{aligned} y_1 &= 1 \\ y_2 &= 2 \\ 3y_1 - 5y_2 + y_3 &= 5 \end{aligned}$$

which has the solution $y_1 = 1$, $y_2 = 2$, $y_3 = 12$.

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 12 \end{bmatrix} \text{ is } \begin{aligned} x_1 + 2x_2 + 2x_3 &= 1 \\ x_2 + 4x_3 &= 2 \\ 17x_3 &= 12 \end{aligned}$$

which gives the solution of the original system:

$$x_1 = \frac{21}{17}, \quad x_2 = -\frac{14}{17}, \quad x_3 = \frac{12}{17}.$$

17. If rows 2 and 3 of A are interchanged, then the resulting matrix has an LU-decomposition. For

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad PA = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix}.$$

Reduce PA to upper triangular form.

$$\begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{3}$, -3, and $\frac{1}{2}$, so

$L = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$. Since $P = P^{-1}$, we have

$$A = PLU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $P\mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$, the system to solve is

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \text{ is } \begin{aligned} 3y_1 \quad &= -2 \\ 2y_2 \quad &= 4 \\ 3y_1 \quad + y_3 &= 1 \end{aligned}$$

which has the solution $y_1 = -\frac{2}{3}$, $y_2 = 2$,

$y_3 = 3$.

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 2 \\ 3 \end{bmatrix} \text{ is}$$

$$x_1 - \frac{1}{3}x_2 \qquad = -\frac{2}{3}$$
$$x_2 + \frac{1}{2}x_3 = 2$$ which gives the solution
$$x_3 = 3$$

to the original system: $x_1 = -\frac{1}{2},\ x_2 = \frac{1}{2},$ $x_3 = 3.$

19. (a) If A has such an LU-decomposition, it can be written as
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x & y \\ wx & wy+z \end{bmatrix}$$
which leads to the equations
$$x = a$$
$$y = b$$
$$wx = c$$
$$wy + z = d$$
Since $a \neq 0$, the system has the unique solution $x = a$, $y = b$, $w = \frac{c}{a}$, and
$$z = d - \frac{bc}{a} = \frac{ad - bc}{a}.$$
Because the solution is unique, the LU-decomposition is also unique.

(b) From part (a) the LU-decomposition is
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & \frac{ad-bc}{a} \end{bmatrix}.$$

True/False 9.1

(a) False; if the matrix cannot be reduced to row echelon form without interchanging rows, then it does not have an LU-decomposition.

(b) False; if the row equivalence of A and U requires interchanging rows of A, then A does not have an LU-decomposition.

(c) True

(d) True

(e) True

Section 9.2

Exercise Set 9.2

1. (a) $\lambda_3 = -8$ is the dominant eigenvalue.

(b) There is no dominant eigenvalue since $|\lambda_1| = |\lambda_4| = 5.$

3. The characteristic polynomial of A is $\lambda^2 - 4\lambda - 6 = 0$, so the eigenvalues of A are $\lambda = 2 \pm \sqrt{10}$. $\lambda = 2 + \sqrt{10} \approx 5.16228$ is the dominant eigenvalue and the dominant eigenvector is $\begin{bmatrix} 1 \\ 3-\sqrt{10} \end{bmatrix} \approx \begin{bmatrix} 1 \\ -0.16228 \end{bmatrix}.$

$$A\mathbf{x}_0 = \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$
$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} = \frac{1}{\sqrt{26}}\begin{bmatrix} 5 \\ -1 \end{bmatrix} \approx \begin{bmatrix} 0.98058 \\ -0.19612 \end{bmatrix}$$
$$A\mathbf{x}_1 \approx \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0.98050 \\ -0.19612 \end{bmatrix} \approx \begin{bmatrix} 5.09902 \\ -0.78446 \end{bmatrix}$$
$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} \approx \begin{bmatrix} 0.98837 \\ -0.15206 \end{bmatrix}$$
$$A\mathbf{x}_2 \approx \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0.98837 \\ -0.15206 \end{bmatrix} \approx \begin{bmatrix} 5.09391 \\ -0.83631 \end{bmatrix}$$
$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} \approx \begin{bmatrix} 0.98679 \\ -0.16201 \end{bmatrix}$$
$$A\mathbf{x}_3 \approx \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0.98679 \\ -0.16201 \end{bmatrix} \approx \begin{bmatrix} 5.09596 \\ -0.82478 \end{bmatrix}$$
$$\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\|A\mathbf{x}_3\|} \approx \begin{bmatrix} 0.98715 \\ -0.15977 \end{bmatrix}$$
$$A\mathbf{x}_4 = \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0.98715 \\ -0.15977 \end{bmatrix} \approx \begin{bmatrix} 5.09552 \\ -0.82738 \end{bmatrix}$$

The dominant unit eigenvector is $\approx \begin{bmatrix} 0.98709 \\ -0.16018 \end{bmatrix}$, which $\mathbf{x}_4$ approximates to three decimal place accuracy.

$$\lambda^{(1)} = (A\mathbf{x}_1)^T\mathbf{x}_1 \approx 5.24297$$
$$\lambda^{(2)} = (A\mathbf{x}_2)^T\mathbf{x}_2 \approx 5.16138$$
$$\lambda^{(3)} = (A\mathbf{x}_3)^T\mathbf{x}_3 \approx 5.16226$$
$$\lambda^{(4)} = (A\mathbf{x}_4)^T\mathbf{x}_4 \approx 5.16223$$

$\lambda^{(4)}$ approximates λ to four decimal place accuracy.

5. The characteristic equation of A is $\lambda^2 - 6\lambda - 4 = 0$, so the eigenvalues of A are $\lambda = 3 \pm \sqrt{13}$.
$\lambda = 3 + \sqrt{13} \approx 6.60555$ is the dominant eigenvalue with corresponding scaled

eigenvector $\begin{bmatrix} \frac{2-\sqrt{13}}{3} \\ 1 \end{bmatrix} \approx \begin{bmatrix} -0.53518 \\ 1 \end{bmatrix}$.

$$A\mathbf{x}_0 = \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda^{(1)} = \frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} = 6$$

$$A\mathbf{x}_1 = \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$$

$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$$

$$\lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} = 6.6$$

$$A\mathbf{x}_2 = \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} -3.5 \\ 6.5 \end{bmatrix}$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \begin{bmatrix} -0.53846 \\ 1 \end{bmatrix}$$

$$\lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} \approx 6.60550$$

$$A\mathbf{x}_3 \approx \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -0.53846 \\ 1 \end{bmatrix} \approx \begin{bmatrix} -3.53846 \\ 6.61538 \end{bmatrix}$$

$$\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\max(A\mathbf{x}_3)} \approx \begin{bmatrix} -0.53488 \\ 1 \end{bmatrix}$$

$$\lambda^{(4)} = \frac{A\mathbf{x}_4 \cdot \mathbf{x}_4}{\mathbf{x}_4 \cdot \mathbf{x}_4} \approx 6.60555$$

7. (a) $A\mathbf{x}_0 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2 \end{bmatrix}$$

$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} = \begin{bmatrix} 1 \\ -0.8 \end{bmatrix}$$

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.6 \end{bmatrix}$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \begin{bmatrix} 1 \\ -0.929 \end{bmatrix}$$

(b) $\lambda^{(1)} = \dfrac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} = 2.8$

$$\lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} \approx 2.976$$

$$\lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} \approx 2.997$$

(c) The characteristic polynomial of A is $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$ so the dominant eigenvalue of A is $\lambda = 3$, with dominant eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(d) The percentage error is $\left|\dfrac{3 - 2.997}{3}\right| = 0.001$ or 0.1%.

9. By Formula (10),

$$\mathbf{x}_5 = \frac{A^5\mathbf{x}_0}{\max(A^5\mathbf{x}_0)} \approx \begin{bmatrix} 0.99180 \\ 1 \end{bmatrix}.$$

Thus $\lambda^{(5)} = \dfrac{A\mathbf{x}_5 \cdot \mathbf{x}_5}{\mathbf{x}_5 \cdot \mathbf{x}_5} \approx 2.99993$.

13. (a) Starting with $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, it takes 8 iterations.

$$\mathbf{x}_1 \approx \begin{bmatrix} 0.229 \\ 0.668 \\ 0.668 \end{bmatrix}, \lambda^{(1)} \approx 7.632$$

$$\mathbf{x}_2 \approx \begin{bmatrix} 0.507 \\ 0.320 \\ 0.800 \end{bmatrix}, \lambda^{(2)} \approx 9.968$$

$$\mathbf{x}_3 \approx \begin{bmatrix} 0.380 \\ 0.197 \\ 0.904 \end{bmatrix}, \lambda^{(3)} \approx 10.622$$

$$\mathbf{x}_4 \approx \begin{bmatrix} 0.344 \\ 0.096 \\ 0.934 \end{bmatrix}, \lambda^{(4)} \approx 10.827$$

$$\mathbf{x}_5 \approx \begin{bmatrix} 0.317 \\ 0.044 \\ 0.948 \end{bmatrix}, \lambda^{(5)} \approx 10.886$$

$$\mathbf{x}_6 \approx \begin{bmatrix} 0.302 \\ 0.016 \\ 0.953 \end{bmatrix}, \lambda^{(6)} \approx 10.903$$

$$\mathbf{x}_7 \approx \begin{bmatrix} 0.294 \\ 0.002 \\ 0.956 \end{bmatrix}, \lambda^{(7)} \approx 10.908$$

$$\mathbf{x}_8 \approx \begin{bmatrix} 0.290 \\ -0.006 \\ 0.957 \end{bmatrix}, \lambda^{(8)} \approx 10.909$$

(b) Starting with $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, it takes 8 iterations.

$$\mathbf{x}_1 \approx \begin{bmatrix} 0.577 \\ 0 \\ 0.577 \\ 0.577 \end{bmatrix}, \lambda^{(1)} \approx 6.333$$

$$\mathbf{x}_2 \approx \begin{bmatrix} 0.249 \\ 0 \\ 0.498 \\ 0.830 \end{bmatrix}, \lambda^{(2)} \approx 8.062$$

$$\mathbf{x}_3 = \begin{bmatrix} 0.193 \\ 0.041 \\ 0.376 \\ 0.905 \end{bmatrix}, \lambda^{(3)} \approx 8.382$$

$$\mathbf{x}_4 \approx \begin{bmatrix} 0.175 \\ 0.073 \\ 0.305 \\ 0.933 \end{bmatrix}, \lambda^{(4)} \approx 8.476$$

$$\mathbf{x}_5 \approx \begin{bmatrix} 0.167 \\ 0.091 \\ 0.266 \\ 0.945 \end{bmatrix}, \lambda^{(5)} \approx 8.503$$

$$\mathbf{x}_6 \approx \begin{bmatrix} 0.162 \\ 0.101 \\ 0.245 \\ 0.951 \end{bmatrix}, \lambda^{(6)} \approx 8.511$$

$$\mathbf{x}_7 \approx \begin{bmatrix} 0.159 \\ 0.107 \\ 0.234 \\ 0.953 \end{bmatrix}, \lambda^{(7)} \approx 8.513$$

$$\mathbf{x}_8 \approx \begin{bmatrix} 0.158 \\ 0.110 \\ 0.228 \\ 0.954 \end{bmatrix}, \lambda^{(8)} \approx 8.513$$

Section 9.3

Exercise Set 9.3

1. The row sums of A are $\mathbf{h}_0 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. The row sums of A^T are $\mathbf{a}_0 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$.

3. $A\mathbf{a}_0 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix}$

$$\mathbf{h}_1 = \frac{A\mathbf{a}_0}{\|A\mathbf{a}_0\|} \approx \begin{bmatrix} 0.39057 \\ 0.65094 \\ 0.65094 \end{bmatrix}$$

$$A^T\mathbf{h}_1 \approx \begin{bmatrix} 1.30189 \\ 0 \\ 1.69246 \end{bmatrix}$$

$$\mathbf{a}_1 = \frac{A^T\mathbf{h}_1}{\|A^T\mathbf{h}_1\|} \approx \begin{bmatrix} 0.60971 \\ 0 \\ 0.79262 \end{bmatrix}$$

5. $A^TA = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The column sums of A are $\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, so we start with

$$\mathbf{a}_0 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix}.$$

$$\mathbf{a}_1 = \frac{A^TA\mathbf{a}_0}{\|A^TA\mathbf{a}_0\|} \approx \begin{bmatrix} 0.70225 \\ 0.70225 \\ 0.11704 \\ 0 \end{bmatrix}$$

$$\mathbf{a}_2 = \frac{A^TA\mathbf{a}_1}{\|A^TA\mathbf{a}_1\|} \approx \begin{bmatrix} 0.70656 \\ 0.70656 \\ 0.03925 \\ 0 \end{bmatrix}$$

$$\mathbf{a}_3 = \frac{A^T A\mathbf{a}_2}{\|A^T A\mathbf{a}_3\|} \approx \begin{bmatrix} 0.70705 \\ 0.70705 \\ 0.01309 \\ 0 \end{bmatrix}$$

$$\mathbf{a}_4 = \frac{A^T A\mathbf{a}_3}{\|A^T A\mathbf{a}_3\|} \approx \begin{bmatrix} 0.70710 \\ 0.70710 \\ 0.00436 \\ 0 \end{bmatrix}$$

$$\mathbf{a}_5 = \frac{A^T A\mathbf{a}_4}{\|A^T A\mathbf{a}_4\|} \approx \begin{bmatrix} 0.70711 \\ 0.70711 \\ 0.00145 \\ 0 \end{bmatrix}$$

$$\mathbf{a}_6 = \frac{A^T A\mathbf{a}_5}{\|A^T A\mathbf{a}_5\|} = \begin{bmatrix} 0.70711 \\ 0.70711 \\ 0.00048 \\ 0 \end{bmatrix}$$

From $\mathbf{a}_6$, it is apparent that sites 1 and 2 are tied for the most authority, while sites 3 and 4 are irrelevant.

7. $A^T A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 2 \end{bmatrix}$

The column sums of A are $\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 2 \end{bmatrix}$, so we start with

$$\mathbf{a}_0 = \frac{1}{\sqrt{19}} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 2 \end{bmatrix} \approx \begin{bmatrix} 0.22942 \\ 0.68825 \\ 0.45883 \\ 0.22942 \\ 0.45883 \end{bmatrix}.$$

$$\mathbf{a}_1 = \frac{A^T A\mathbf{a}_0}{\|A^T A\mathbf{a}_0\|} \approx \begin{bmatrix} 0.15250 \\ 0.71166 \\ 0.55916 \\ 0.30500 \\ 0.25416 \end{bmatrix}$$

$$\mathbf{a}_2 = \frac{A^T A\mathbf{a}_1}{\|A^T A\mathbf{a}_1\|} \approx \begin{bmatrix} 0.08327 \\ 0.72861 \\ 0.58289 \\ 0.32267 \\ 0.13531 \end{bmatrix}$$

$$\mathbf{a}_3 = \frac{A^T A\mathbf{a}_2}{\|A^T A\mathbf{a}_2\|} \approx \begin{bmatrix} 0.04370 \\ 0.73455 \\ 0.58889 \\ 0.32670 \\ 0.07075 \end{bmatrix}$$

$$\mathbf{a}_4 = \frac{A^T A\mathbf{a}_3}{\|A^T A\mathbf{a}_3\|} \approx \begin{bmatrix} 0.02273 \\ 0.73630 \\ 0.59045 \\ 0.32766 \\ 0.03677 \end{bmatrix}$$

$$\mathbf{a}_5 = \frac{A^T A\mathbf{a}_4}{\|A^T A\mathbf{a}_4\|} \approx \begin{bmatrix} 0.01179 \\ 0.73679 \\ 0.59086 \\ 0.32790 \\ 0.01908 \end{bmatrix}$$

$$\mathbf{a}_6 = \frac{A^T A\mathbf{a}_5}{\|A^T A\mathbf{a}_5\|} \approx \begin{bmatrix} 0.00612 \\ 0.73693 \\ 0.59097 \\ 0.32796 \\ 0.00990 \end{bmatrix}$$

From $\mathbf{a}_6$, it is apparent that site 2 has the most authority, followed by sites 3 and 4, while sites 1 and 5 are irrelevant.

Section 9.4

Exercise Set 9.4

1. **(a)** For $n = 1000 = 10^3$, the flops for both phases is
$\frac{2}{3}(10^3)^3 + \frac{3}{2}(10^3)^2 - \frac{7}{6}(10^3) = 668,165,500,$
which is 0.6681655 gigaflops, so it will take
$0.6681655 \times 10^{-1} \approx 0.067$ second.

(b) $n = 10,000 = 10^4$:
$\frac{2}{3}(10^4)^3 + \frac{3}{2}(10^4)^2 - \frac{7}{6}(10^4)$
$= 666,816,655,000$ flops
or 666.816655 gigaflops
The time is about 66.68 seconds.

(c) $n = 100,000 = 10^5$
$\frac{2}{3}(10^5)^3 + \frac{3}{2}(10^5)^2 - \frac{7}{6}(10^5)$
$\approx 666,682 \times 10^9$ flops
or 666,682 gigaflops
The time is about 66,668 seconds which is about 18.5 hours.

3. $n = 10{,}000 = 10^4$

(a) $\frac{2}{3}n^3 \approx \frac{2}{3}(10^{12}) \approx 666.67\times 10^9$
666.67 gigaflops are required, which will take $\frac{666.67}{70} \approx 9.52$ seconds.

(b) $n^2 \approx 10^8 = 0.1\times 10^9$
0.1 gigaflop is required, which will take about 0.0014 second.

(c) This is the same as part (a); about 9.52 seconds.

(d) $2n^3 \approx 2\times 10^{12} = 2000\times 10^9$
2000 gigaflops are required, which will take about 28.57 seconds.

5. (a) $n = 100{,}000 = 10^5$

$$\begin{aligned}\frac{2}{3}n^3 &\approx \frac{2}{3}\times 10^{15}\\ &\approx 0.667\times 10^{15}\\ &= 6.67\times 10^5\times 10^9\end{aligned}$$

Thus, the forward phase would require about 6.67×10^5 seconds.
$n^2 = 10^{10} = 10\times 10^9$
The backward phase would require about 10 seconds.

(b) $n = 10{,}000 = 10^4$

$$\begin{aligned}\frac{2}{3}n^3 &\approx \frac{2}{3}\times 10^{12}\\ &\approx 0.667\times 10^{12}\\ &\approx 6.67\times 10^2\times 10^9\end{aligned}$$

About 667 gigaflops are required, so the computer would have to execute 2(667) = 1334 gigaflops per second.

7. Multiplying each of the n^2 entries of A by c requires n^2 flops.

9. Let $C = [c_{ij}] = AB$. Computing c_{ij} requires first multiplying each of the n entries a_{ik} by the corresponding entry b_{kj}, which requires n flops. Then the n terms $a_{ik}b_{kj}$ must be summed, which requires $n - 1$ flops. Thus, each of the n^2 entries in AB requires $2n - 1$ flops, for a total of $n^2(2n-1) = 2n^3 - n^2$ flops.
Note that adding two numbers requires 1 flop, adding three numbers requires 2 flops, and in general, $n - 1$ flops are required to add n numbers.

Section 9.5

Exercise Set 9.5

1. The characteristic polynomial of
$A^TA = \begin{bmatrix}1\\2\\0\end{bmatrix}[1 \quad 2 \quad 0] = \begin{bmatrix}1&2&0\\2&4&0\\0&0&0\end{bmatrix}$ is $\lambda^2(\lambda-5)$; thus the eigenvalues of A^TA are $\lambda_1 = 5$ and $\lambda_2 = 0$, and $\sigma_1 = \sqrt{5}$ and $\sigma_2 = 0$ are singular values of A.

3. The eigenvalues of
$A^TA = \begin{bmatrix}1&2\\-2&1\end{bmatrix}\begin{bmatrix}1&-2\\2&1\end{bmatrix} = \begin{bmatrix}5&0\\0&5\end{bmatrix}$ are $\lambda_1 = 5$ and $\lambda_2 = 5$ (i.e., $\lambda = 5$ is an eigenvalue of multiplicity 2); thus the singular value of A is $\sigma_1 = \sqrt{5}$.

5. The only eigenvalue of
$A^TA = \begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}1&-1\\1&1\end{bmatrix} = \begin{bmatrix}2&0\\0&2\end{bmatrix}$ is $\lambda = 2$ (multiplicity 2), and the vectors $\mathbf{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ form an orthonormal basis for the eigenspace (which is all of R^2). The singular values of A are $\sigma_1 = \sqrt{2}$ and $\sigma_2 = \sqrt{2}$. We have
$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$, and
$\mathbf{u}_2 = \frac{1}{\sigma_2}A\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$. This results in the following singular value decomposition of A:

$$A = U\Sigma V^T = \begin{bmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\sqrt{2}&0\\0&\sqrt{2}\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

7. The eigenvalues of
$A^TA = \begin{bmatrix} 4 & 0 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 24 \\ 24 & 52 \end{bmatrix}$ are $\lambda_1 = 64$ and $\lambda_2 = 4$, with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ respectively. The singular values of A are $\sigma_1 = 8$ and $\sigma_2 = 2$. We have

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{8}\begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \text{ and}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2}A\mathbf{v}_2 = \frac{1}{2}\begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}.$$

This results in the following singular value decomposition:

$$A = U\Sigma V^T = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{15}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

9. The eigenvalues of
$A^TA = \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$ are $\lambda_1 = 18$ and $\lambda_2 = 0$, with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ respectively. The only nonzero singular value of A is $\sigma_1 = \sqrt{18} = 3\sqrt{2}$, and we have

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{3\sqrt{2}}\begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}.$$

We must choose the vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ so that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis R^3.

A possible choice is $\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and

$\mathbf{u}_3 = \begin{bmatrix} \frac{\sqrt{2}}{6} \\ -\frac{2\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{6} \end{bmatrix}$. This results in the following singular value decomposition:

$$A = U\Sigma V^T = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6} \\ \frac{1}{3} & 0 & -\frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \end{bmatrix}\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Note: The singular value decomposition is not unique. It depends on the choice of the (extended) orthonormal basis for R^3. This is just one possibility.

11. The eigenvalues of
$A^TA = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ are $\lambda_1 = 3$ and $\lambda_2 = 2$, with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively. The singular values of A are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = \sqrt{2}$. We have

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} \text{ and}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}. \text{ We choose}$$

$\mathbf{u}_3 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ so that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for R^3. This results in the following singular value decomposition:

$$A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

True/False 9.5

(a) False; if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, and $A^T A$ is an $n \times n$ matrix.

(b) True

(c) False; $A^T A$ may have eigenvalues that are 0.

(d) False; A would have to be symmetric to be orthogonally diagonalizable.

(e) True; since $A^T A$ is a symmetric $n \times n$ matrix.

(f) False; the eigenvalues of $A^T A$ are the squares of the singular values of A.

(g) True

Section 9.6

Exercise Set 9.6

1. From Exercise 9 in Section 9.5, A has the singular value decomposition

$$A = \left[\begin{array}{c|cc} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6} \\ \frac{1}{3} & 0 & -\frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \end{array}\right]\left[\begin{array}{c|c} 3\sqrt{2} & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \hline \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right]$$

where the lines show the relevant blocks. Thus the reduced singular value decomposition of A is

$$A = U_1\Sigma_1V_1^T = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}\left[3\sqrt{2}\right]\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

3. From Exercise 11 in Section 9.5, A has the singular value decomposition

$$A = \left[\begin{array}{cc|c} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{array}\right]\left[\begin{array}{cc} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ \hline 0 & 0 \end{array}\right]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where the lines indicate the relevant blocks. Thus the reduced singular value decomposition of A is

$$A = U_1\Sigma_1V_1^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

5. The reduced singular value expansion of A is

$$3\sqrt{2}\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

7. The reduced singular value decomposition of A is $\sqrt{3}\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}[1 \;\; 0] + \sqrt{2}\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}[0 \;\; 1].$

9. A rank 100 approximation of A requires storage space for 100(200 + 500 + 1) = 70,100 numbers, while A has 200(500) = 100,000 has entries.

True/False 9.6

(a) True

(b) True

(c) False; V_1 has size $n \times k$ so that V_1^T has size $k \times n$.

Chapter 9 Supplementary Exercises

1. Reduce A to upper triangular form.

$$\begin{bmatrix} -6 & 2 \\ 6 & 0 \end{bmatrix} \to \begin{bmatrix} -3 & 1 \\ 6 & 0 \end{bmatrix} \to \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{2}$ and 2, so

$$L = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}.$$

3. Reduce A to upper triangular form.

$$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 4 & 7 \\ 1 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \\ 1 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{2}$, $-1, -1$, $\frac{1}{2}$, -1, and $\frac{1}{2}$ so $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. (a) The characteristic equation of A is $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$ so the dominant eigenvalue of A is $\lambda_1 = 3$, with corresponding positive unit eigenvector

$$\mathbf{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \approx \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}.$$

(b) $A\mathbf{x}_0 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.8944 \\ 0.4472 \end{bmatrix}$$

$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} \approx \begin{bmatrix} 0.7809 \\ 0.6247 \end{bmatrix}$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} \approx \begin{bmatrix} 0.7328 \\ 0.6805 \end{bmatrix}$$

$$\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\|A\mathbf{x}_3\|} \approx \begin{bmatrix} 0.7158 \\ 0.6983 \end{bmatrix}$$

$$\mathbf{x}_5 = \frac{A\mathbf{x}_4}{\|A\mathbf{x}_4\|} \approx \begin{bmatrix} 0.7100 \\ 0.7042 \end{bmatrix}$$

$\mathbf{x}_5 \approx \begin{bmatrix} 0.7100 \\ 0.7042 \end{bmatrix}$ as compared to

$\mathbf{v} \approx \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}.$

(c) $A\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \begin{bmatrix} 1 \\ 0.9286 \end{bmatrix}$$

$$\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\max(A\mathbf{x}_3)} \approx \begin{bmatrix} 1 \\ 0.9756 \end{bmatrix}$$

$$\mathbf{x}_5 = \frac{A\mathbf{x}_4}{\max(A\mathbf{x}_4)} \approx \begin{bmatrix} 1 \\ 0.9918 \end{bmatrix}$$

$\mathbf{x}_5 \approx \begin{bmatrix} 1 \\ 0.9918 \end{bmatrix}$ as compared to the exact eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

7. The Rayleigh quotients will converge to the dominant eigenvalue $\lambda_4 = -8.1$. However, since the ratio $\frac{|\lambda_4|}{|\lambda_1|} = \frac{8.1}{8} = 1.0125$ is very close to 1, the rate of convergence is likely to be quite slow.

9. The eigenvalues of $A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ are $\lambda_1 = 4$ and $\lambda_2 = 0$ with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, respectively. The only nonzero singular value A is $\sigma_1 = \sqrt{4} = 2$, and we have

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

We must choose the vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ so that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for R^3. A possible choice is $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.

This results in the following singular value decomposition:

$$A = U\Sigma V^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

11. A has rank 2, thus $U_1 = [\mathbf{u}_1 \quad \mathbf{u}_2]$ and $V_1^T = \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix}$ and the reduced singular value decomposition of A is

$$A = U_1 \Sigma_1 V_1^T$$
$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 24 & 0 \\ 0 & 12 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$