① $c_1 u_1 + c_2 u_2 + c_3 u_3 = v$

$$\begin{array}{c} x_1\text{'s} \\ x_2\text{'s} \\ x_3\text{'s} \\ \\ \end{array}\begin{array}{c}\left[\begin{array}{ccc|c} 3 & 4 & -6 & 41 \\ 3 & 4 & 3 & 5 \\ 6 & 3 & 3 & 12 \end{array}\right] \\ \begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$$\boxed{v = 3u_1 + 2u_2 - 4u_3}$$

---

② $c_1 u_1 + c_2 u_2 + c_3 u_3 = v$

$$\begin{array}{c}\left[\begin{array}{ccc|c} 3 & 4 & -3 & 4 \\ 5 & 2 & 2 & 9 \\ 2 & 4 & -3 & 3 \\ 10 & 4 & 4 & 8 \end{array}\right] \\ \begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \Leftarrow \text{inconsistent}$$

$$\boxed{\text{IMPOSSIBLE}}$$

---

③ $\boxed{\text{Yes}}$, all M_{nm} are vector spaces.

---

④ $V = \langle a, b, c, d, c \rangle$, $a \neq 0$

$\boxed{\text{No,}}$ not a vector space. There does not exist a fourth degree polynomial $z(x)$ in V such that $p(x) + z(x) = p(x)$. (Due to the restriction $a \neq 0$)

---

⑤ $V = \{(x, x, x) \mid x \in \mathbb{R}\}$

$(x_1, x_1, x_1) + (x_2, x_2, x_2) = (x_1 + x_2, x_1 + x_2, x_1 + x_2) \in V$ ✓ closed under addition

$k(x_1, x_1, x_1) = (kx_1, kx_1, kx_1) \in V$ ✓ closed under ~~the~~ scalar multiplication

V is a subspace of $\mathbb{R}^3$, and $\mathbb{R}^3$ is a vector space, $\therefore$ $\boxed{V \text{ is a vector space.}}$

Note
the symbol $\in$ is short for "is in". i.e: $(2,4)\in\mathbb{R}^2$.

(6) V = all matrices of the form $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & 0 & 0 \\ 0 & b_1+b_2 & 0 \\ 0 & 0 & c_1+c_2 \end{bmatrix} \in V$$ ✓ closed under addition

$$k\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} ka & 0 & 0 \\ 0 & kb & 0 \\ 0 & 0 & kc \end{bmatrix} \in V$$ ✓ closed under scalar multiplication

V is a subspace of M_3, and M_3 is a vector space.

$\therefore$ $\boxed{V \text{ is a vector space.}}$

---

(7) $W = \{(x, y, 5x-7y) \mid (x,y) \in \mathbb{R}\}$

$$(x_1, y_1, 5x_1-7y_1) + (x_2, y_2, 5x_2-7y_2) = (x_1+x_2, y_1+y_2, 5x_1-7y_1+5x_2-7y_2)$$
$$= (x_1+x_2, y_1+y_2, 5(x_1+x_2)-7(y_1+y_2)) \in W$$ ✓ c.u.a.

$$k(x,y,5x-7y) = (kx, ky, k(5x-7y)) = (kx, ky, 5(kx)-7(ky)) \in W$$ ✓ c.u.s.m.

$\boxed{W \text{ is a subspace of } \mathbb{R}^3.}$

---

(8) V = all matrices of the form $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$

$$\begin{bmatrix} a_{111} & \cdots & a_{1n1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn1} \end{bmatrix} + \begin{bmatrix} a_{112} & \cdots & a_{1n2} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn2} \end{bmatrix} = \begin{bmatrix} a_{111}+a_{112} & \cdots & a_{1n1}+a_{1n2} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn1}+a_{nn2} \end{bmatrix} \in V$$ ✓ c.u.a.

$$k\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & ka_{nn} \end{bmatrix} \in V$$ ✓ c.u.s.m.

$\boxed{V \text{ is a subspace of } M_{nn}.}$

(9) $(0, x_{21}, a, x_{41}) + (0, x_{22}, a, x_{42}) = (0, x_{21}+x_{22}, 18, x_{41}+x_{42}) \notin W$ X not c.u.a.

$\boxed{\text{FALSE}}$

---

(10) $\begin{vmatrix} 6 & -5 \\ 5 & 6 \end{vmatrix} = 61 \therefore \boxed{S \text{ spans } \mathbb{R}^2}$

$\boxed{\text{TRUE}}$

explanation

$c_1\langle 6,5\rangle + c_2\langle -5,6\rangle = \langle x,y\rangle$

For any $\langle x,y\rangle$ (spanning $\mathbb{R}^2$).

This can be rewritten as

$$\begin{bmatrix} 6 & -5 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Checking that the determinant $\neq 0$ verifies that this system can be solved by inverting the matrix:

$$Ax = d \rightarrow x = A^{-1}d$$

---

(11) $\boxed{\text{FALSE}}$. At least 2 vectors needed to span $\mathbb{R}^2$.

---

(12) $\begin{vmatrix} -1 & 3 \\ 5 & -15 \end{vmatrix} = 0 \therefore \boxed{\text{FALSE}}$

S does not span $\mathbb{R}^2$.

---

(13) $c_1(-6,7) + c_2(7,-6) = (1,1)$

Remove $(1,1)$ from S.

$S_1 = \{(-6,7) + (7,-6)\}$

Note: We can remove $(1,1)$ because it is a Linear Combination of the other vectors. now all of our solutions could be written like:
$c_1\langle -6,7\rangle + c_2\langle 7,-6\rangle + 0\langle 1,1\rangle = \langle x,y\rangle$

$\begin{vmatrix} -6 & 7 \\ 7 & -6 \end{vmatrix} = -13 \therefore \boxed{\text{TRUE}}$, S spans $\mathbb{R}^2$

---

(14) $\begin{vmatrix} 9 & 9 & 0 \\ 0 & 9 & 9 \\ 9 & 0 & 9 \end{vmatrix} = 1458 \therefore \boxed{\text{TRUE}}$

S spans $\mathbb{R}^3$.

---

(15) $\left[\begin{array}{cc|c} -5 & 4 & 0 \\ 6 & 7 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \therefore \boxed{\text{TRUE}}$

S is L.I.

---

(16) $\left[\begin{array}{ccc|c} 1 & 2 & 7 & 0 \\ 1 & 2 & 7 & 0 \\ 1 & 2 & 7 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 2 & 7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \therefore \boxed{\text{FALSE}}$

S is not L.I.

<u>Determining Linear Independence.</u>

A set of vectors $S = \{v_1, v_2, \cdots, v_n\}$ should satisfy the equation:

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}.$$

One sol'n: $c_1 = 0, c_2 = 0, \cdots, c_n = 0$

called the trivial sol'n

If another sol'n exists, the set is called <u>linearly dependent.</u>

If only the trivial sol'n exists, the set is called <u>linearly dependent.</u>

(17) $c_1(2,3) + c_2(4,6) = (26,39)$

$$\left[\begin{array}{cc|c} 2 & 4 & 26 \\ 3 & 6 & 39 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \end{array}\right]$$

$3\langle 2,3\rangle + 5\langle 4,6\rangle = \langle 26,39\rangle$

---

(18)
$$\begin{array}{c} x^0 \\ x^1 \\ x^2 \end{array} \begin{vmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 4 \therefore \boxed{\text{TRUE}}$$

S spans P_2.

---

(19) $\{1, x, x^2, x^3, x^4\}$

---

(20) $\begin{vmatrix} 8 & 16 \\ -3 & -6 \end{vmatrix} = 0$, so the matrix is not invertible.

OR

By inspection: $2(8,-3) = (16,-6)$, so the set is linearly dependent and therefore not a basis.

---

(21) $\begin{vmatrix} 2 & -2 & 8 \\ 1 & -1 & 4 \\ -3 & 3 & -12 \end{vmatrix} = 0$, so the matrix is not invertible.

OR

$$\left[\begin{array}{ccc|c} 2 & -2 & 8 & 0 \\ 1 & -1 & 5 & 0 \\ -3 & 3 & -12 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \underbrace{\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}$$

there exists a nontrivial sol'n, so the set is linearly dependent.

---

(22) $\begin{vmatrix} 0 & 8 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{vmatrix} = 0$, so the matrix is not invertible.

OR

$$\left[\begin{array}{ccc|c} 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \underbrace{\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}$$

there exists a nontrivial sol'n, so the set is linearly dependent.

---

(23) $$\left[\begin{array}{ccc|c} 8 & 0 & 0 & 0 \\ 7 & 7 & 0 & 0 \\ 6 & 6 & 6 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \therefore \text{L.I.}$$

$\begin{vmatrix} 8 & 0 & 0 \\ 7 & 7 & 0 \\ 6 & 6 & 6 \end{vmatrix} = 336 \therefore$ spans $\mathbb{R}^3$

$\therefore S$ is a basis for $\mathbb{R}^3$.

$c_1\langle 8,7,6\rangle + c_2\langle 0,7,6\rangle + c_3\langle 0,0,6\rangle = \langle 24,14,36\rangle$

$$\left[\begin{array}{ccc|c} 8 & 0 & 0 & 24 \\ 7 & 7 & 0 & 14 \\ 6 & 6 & 6 & 36 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$3\langle 8,7,6\rangle - 1\langle 0,7,6\rangle + 4\langle 0,0,6\rangle = \langle 24,14,36\rangle$

24 $\begin{vmatrix} 0 & 1 & 8 \\ 0 & 3 & 1 \\ 0 & 6 & -3 \end{vmatrix} = 0 \therefore$ $\boxed{\text{not a basis for } \mathbb{R}^3}$

(S does not span $\mathbb{R}^3$)

---

25 $\begin{vmatrix} 5/6 & 1 & 12 \\ 9/2 & 9/2 & 63 \\ 2 & 0 & 24 \end{vmatrix} = 0 \therefore$ $\boxed{\text{not a basis for } \mathbb{R}^3}$

(S does not span $\mathbb{R}^3$)

---

26 $\begin{bmatrix} -18 & -3 & 3 & 26 \\ 18 & 3 & -3 & -24 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & \frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

basis for row space: (r_1, r_2)

basis for col space: $\underbrace{(c_1, c_4)}$

same size $\rightarrow$ $\boxed{\text{rank} = 2}$

---

27 $\left[\begin{array}{ccc|c} 42 & 54 & 72 & 0 \\ 14 & 18 & 24 & 0 \\ 7 & 9 & 8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & \frac{9}{7} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = -\frac{9}{7}t$

$x_2 = t$

$x_3 = s$

$\vec{x} = s\langle 0,0,1\rangle + t\langle -\frac{9}{7}, 1, 0\rangle$

$\boxed{\{(0,0,1), (-9,7,0)\}}$

* NOTE: When finding a basis, you can multiply any of the components by a scalar.

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28 $\underbrace{\begin{bmatrix} 7 & 0 \\ -2 & 1 \end{bmatrix}}_{\text{Basis}} \underbrace{\begin{bmatrix} 8 \\ 1 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} 56 \\ -15 \end{bmatrix}$ $\boxed{(56, -15)}$

TRANSITION MATRICES

To find a transition matrix from basis A to basis B, set up a matrix and RREF:

$[B \mid A] \rightarrow [I \mid T_{A\to B}]$

---

29 $\left[\begin{array}{cc|cc} 1 & 0 & 7 & -6 \\ 0 & 1 & 5 & 9 \end{array}\right]$ with C = left block, B = right block (already in RREF) $\boxed{T_{B\to C} \begin{bmatrix} 7 & -6 \\ 5 & 9 \end{bmatrix}}$

---

30 $\left[\begin{array}{ccc|c} 5 & 8 & 1 & 5 \\ 11 & 0 & 2 & 28 \\ 0 & 8 & 7 & 13 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$ $\boxed{[x]_B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}}$

---

31 Basis for M_{31}: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ $\boxed{[x]_S = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}}$

15c $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\boxed{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$

17a $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

$\boxed{(-2, 1, 0)}$

19b $\begin{bmatrix} \cos 45 & 0 & \sin 45 \\ 0 & 1 & 0 \\ -\sin 45 & 0 & \cos 45 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

$\boxed{(0, 1, 2\sqrt{2})}$