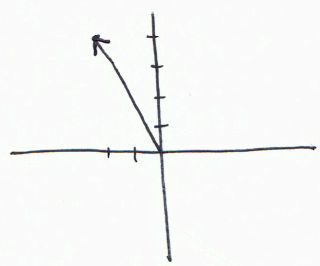
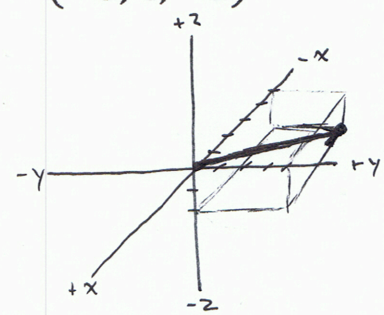
Overview

- This is a vector: $\vec{v} = (x_1, x_2, \dots, x_n)$
- A vector looks like a coordinate. You can tell the difference by the arrow above the variable.
- The different values in a vector are called components.
- A vector in n-space (R^n) has n components. The above vector is in n-space.
    - 2-space: $(2, -3)$
    - 4-space: $\left(-1, 4, 12, \frac{12}{3}\right)$
- A vector can also be written as a sum of components.
    - $\vec{v} = (-3, 6) = -3\hat{\imath} + 6\hat{\jmath}$
    - $\vec{w} = \left(\frac{3}{2}, 3, -\frac{2}{3}\right) = -\frac{3}{2}\hat{\imath} + 3\hat{\jmath} - \frac{2}{3}\hat{k}$
- You can visualize a vector in 2 or 3-space as an arrow from the origin to the coordinate in the vector:

$(-2, 4)$



$(-5, 4, -2)$



3.1 – Vectors in 2-space, 3-space, and n-space

- Displacement vector from A to B:

A (3, 6) ★ final − initial

B (−2, 12)

$\vec{AB}\ (-2-3, 12-6) = (-5, 6)$

A (−2, 4)

B (0, 4)

$\vec{AB} = (0-(-2), 4-4) = (2, 0)$

- Magnitude (norm) of a vector: $\boxed{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$ ★ written $\|\vec{v}\|$

$\vec{V} = \langle 3, -2, 7 \rangle \quad \|\vec{v}\| = \sqrt{9+4+49} = \sqrt{62}$

$\vec{V} = \langle 1, 2, 3, 4 \rangle \quad \|\vec{v}\| = \sqrt{1+2+9+16} = \sqrt{28} = 2\sqrt{7}$

- Addition/subtraction

★ add each component separately

$\langle 2,-3\rangle + \langle -1,4\rangle = \langle 2+(-1), -3+4\rangle = \langle 1,1\rangle$

$\langle 1,2,-3,4,-5\rangle + \langle 6,8,0,10,-1\rangle = \langle 7,10,-3,14,-6\rangle$

- Scalar multiplication

★ multiply every component by the scalar

$2\langle 3,4\rangle = \langle 6,8\rangle$ | $x^2\langle 3,x,5\rangle = \langle 3x^2, x^3, 5x^2\rangle$

$4\langle \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\rangle = \langle 1,2,3\rangle$ | $-\frac{1}{2}\langle 3,10,-2\rangle = \langle -\frac{3}{2}, -5, 1\rangle$

3.2 – Norm, Dot Product, and Distance in R^n

- The distance between two points is the norm of their displacement vector.

Distance between $\underset{A}{(4,-8)}$ and $\underset{B}{(-3,25)}$

$\vec{v} = (-3-4, 25-(-8)) = (-7,33)$ ← vector from A to B

$\|\vec{v}\| = \sqrt{7^2+33^2} = \sqrt{1138}$

- $\boxed{\text{A unit vector has length 1.}}$
- To find a unit vector in the same direction as another vector, divide each component by the magnitude of the vector.

Unit vector in same direction as $\langle -2,6,-9\rangle$

$\|\vec{v}\| = \sqrt{4+36+81} = \sqrt{121} = 11$

$\vec{u} = \langle -\frac{2}{11}, \frac{6}{11}, -\frac{9}{11}\rangle$ ← has magnitude 1

"unit" vector

- Standard basis vectors in n-space:

$\hat{i} = \langle 1,0,0\rangle$

$\hat{j} = \langle 0,1,0\rangle$

$\hat{k} = \langle 0,0,1\rangle$

- Dot product of two vectors

$\vec{A} = \langle 2, -3, 9 \rangle$ $\vec{B} = \langle 0, 1, 10 \rangle$ $\vec{C} = \langle -3, 7, 2 \rangle$

$\vec{A} \cdot \vec{B}$

$(2)(0)$ x_1
$+(-3)(1)$ x_2
$+(9)(10)$ x_3

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$\vec{B} \cdot \vec{C}$

$(0)(-3) + (1)(7) + (10)(2) = 27$

x_1 x_2 x_3

$\vec{A} \cdot \vec{C}$ $\vec{A} \cdot \vec{C}$

$(2)(-3) + (-3)(7) + (9)(2) = -6 - 21 + 18 = -9$

x_1 x_2 x_3

- $\boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta}$
- θ is the angle between the vectors.
- Angle between vectors:

$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right)}$$ ☆ angle between vectors

$\vec{a} = \langle 2, -11 \rangle$ $\vec{a} \cdot \vec{b} = 2(4) + (-11)(5) = -47$ $\|\vec{a}\| = \sqrt{4, 121} = \sqrt{125} = 5\sqrt{5}$

$\vec{b} = \langle 4, 5 \rangle$ $\|\vec{b}\| = \sqrt{16 + 25} = \sqrt{41}$

The angle between $\vec{a}$ and $\vec{b} = \cos^{-1}\left(\frac{-47}{(5\sqrt{5})(\sqrt{41})}\right) = 131°$

$\vec{a} = \langle 1, 2, 3 \rangle$ $\vec{a} \cdot \vec{b} = 1(1) + 2(-1) + 3(1) = 2$ $\|\vec{a}\| = \sqrt{1 + 4 + 9} = \sqrt{14}$

$\vec{b} = \langle 1, -1, 1 \rangle$ $\|\vec{b}\| = \sqrt{1 + 1 + 1} = \sqrt{3}$

Angle between $\vec{a}$ and $\vec{b} = \cos^{-1}\left(\frac{2}{\sqrt{14}\sqrt{3}}\right) = 72.02°$

3.3 – Orthogonality

- Angle between parallel vectors = 180° or 0°

☆ Dot product equals $\|\vec{a}\|\|\vec{b}\|$ or $-\|\vec{a}\|\|\vec{b}\|$

$\vec{A}\langle 4, -8, 6 \rangle$ $\vec{A} \cdot \vec{B} = 4(-2) + (-8)(4) + 6(-3) = -58$ Parallel

$\vec{B}\langle -2, 4, -3 \rangle$ $\|\vec{A}\| = \sqrt{16 + 64 + 36} = \sqrt{116}$ $(\sqrt{116})(\sqrt{29}) = 58$

$\|\vec{B}\| = \sqrt{4 + 16 + 9} = \sqrt{29}$

- Angle between perpendicular (orthogonal) vectors = 90°

☆ Dot product is 0

$\vec{A}\langle 2, 4, 2 \rangle$ $\vec{A} \cdot \vec{B} = 2(13) + 4(-6) + 2(-1) = 0$

$\vec{B}\langle 13, -6, -1 \rangle$

$\vec{A}$ and $\vec{B}$ are orthogonal.

- Vector component of vector $\vec{u}$ along $\vec{a}$:

$$\boxed{\underbrace{\left(\frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\right)}_{\text{scalar}}\vec{a}}$$

component of $\underset{\vec{u}}{(1,-2)}$ along $\underset{\vec{a}}{(-4,-3)}$

$$\left(\frac{2}{25}\right)\langle -4,-3\rangle = \left\langle -\frac{8}{25}, -\frac{6}{25}\right\rangle$$

component of $\underset{\vec{u}}{(3,0,4)}$ along $\underset{\vec{a}}{(2,3,3)}$

$$\left(\frac{18}{22}\right)\langle 2,3,3\rangle = \left\langle \frac{18}{11}, \frac{27}{11}, \frac{27}{11}\right\rangle$$

- Vector component of vector $\vec{u}$ orthogonal to $\vec{a}$:

$$\boxed{\vec{u}-\left(\frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\right)\vec{a}}$$

component of $\underset{\vec{u}}{(1,-2)}$ orthogonal to $\underset{\vec{a}}{(-4,-3)}$

$$\langle 1,-2\rangle - \underbrace{\left\langle -\frac{8}{25}, -\frac{6}{25}\right\rangle}_{\text{result from above}} = \left\langle \frac{33}{25}, \frac{-44}{25}\right\rangle$$

component of $\underset{\vec{u}}{(3,0,4)}$ orthogonal to $\underset{\vec{a}}{(2,3,3)}$

$$\langle 3,0,4\rangle - \underbrace{\left\langle \frac{18}{11}, \frac{27}{11}, \frac{27}{11}\right\rangle}_{\text{result from above}} =$$

3.4 – Geometry of Linear Systems

- Solutions of non-homogenous systems are found by adding what to the solutions of homogeneous systems?
- Solve both systems and find the difference. It will likely require parameters.
- Answer = (non-homogeneous solution) – (homogeneous solution)

★ Be sure to use the same parameters in both solutions: if $x_4 = t$ in one solution, and you need to parameterize x_4 in the second solution, you must also use t.

3.5 – Cross Product

- The cross product of two vectors is written: $\vec{a}\times\vec{b}$.
- The cross product will give a vector orthogonal to both $\vec{a}$ and $\vec{b}$.
- How to calculate: ★ Like determinant with $\hat{i}$, $\hat{j}$, and $\hat{k}$.

$\vec{a} = (-2, 4, 5)$ $\vec{b} = (3, -7, 1)$

$$\left|\begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ -2 & 4 & 5 & -2 & 4 \\ 3 & -7 & 1 & 3 & -7 \end{array}\right.$$

$+\hat{i}(4)(1) + \hat{j}(5)(3) + \hat{k}(-2)(-7)$

$-\hat{k}(4)(3) - \hat{i}(5)(-7) - \hat{j}(-2)(1)$

$39\hat{i} + 17\hat{j} + 2\hat{k}$

$\langle 39, 17, 2\rangle = \vec{a}\times\vec{b}$

- To find two vectors orthogonal to both $\vec{a}$ and $\vec{b}$, use:
  - $\vec{a}\times\vec{b}$
  - $-(\vec{a}\times\vec{b})$ (This is equal to $\vec{b}\times\vec{a}$.)

$\vec{a} = (-2, 4, 5)$

$\vec{b} = (3, -7, 1)$

$$\left.\begin{aligned} \vec{a}\times\vec{b} &= \langle 39, 17, 2\rangle \\ -(\vec{a}\times\vec{b}) &= \langle -39, -17, 2\rangle \end{aligned}\right\}\ \text{both orthogonal to } \vec{a} \text{ and } \vec{b}$$