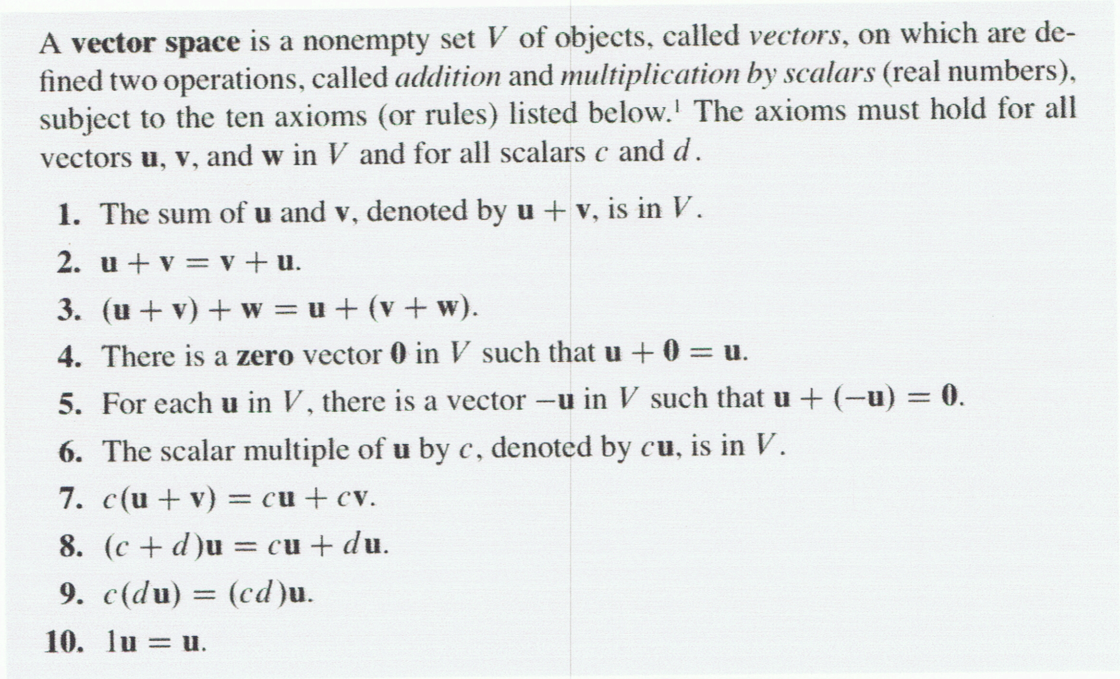
Overview

- A vector space is a set of vectors that satisfies a certain set of rules (4.1).
- A set is a collection of vectors of a certain form, and a set of rules for addition and subtraction of those vectors.

4.1 – Real Vector Spaces

- A set of vectors is a vector space if it the below ten axioms are true for any vectors **u** and **v** in that set of vectors.



A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

- You can remember **CAIN** for addition and **DDAI** for multiplication:
  - **Closed Under Addition**
  - Commutative
  - Associative
  - Identity
  - Negative
  - **Closed Under Scalar Multiplication**
  - Distributive
  - Distributive
  - Associative
  - Identity

- To show that a set is a vector space, you need to prove all ten of those axioms. Before you begin, choose a vector **u** and a vector **v** to use for your proofs.
- Do this with variables. Suppose this is your set:
  - The set of all triples of the form $\langle x, y, x^2\rangle$ with the standard operations on R^3.
  - You can let $\vec{u} = \langle x_1, y_1, x_1^2\rangle$
  - And let $\vec{v} = \langle x_2, y_2, x_2^2\rangle$

- The first axioms you should check are 1 and 6. Using the above example, we can show that this set is not closed under addition or multiplication:

ADDITION:

$$\vec{u} + \vec{v} = \langle x_1, y_1, x_1^2\rangle + \langle x_2, y_2, x_2^2\rangle = \langle x_1+x_2, y_1+y_2, \overbrace{x_1^2+x_2^2}^{\text{bad!}}\rangle$$

This result is not part of the set, because

$$\underbrace{(x_1+x_2)^2}_{\text{what the third component should be}} \neq \left\{x_1^2+x_2^2\right\} \text{— what the third component is}$$

MULTIPLICATION:

$$c\vec{u} = c\langle x_1, y_1, x_1^2\rangle = \langle cx_1, cy_1, \overbrace{cx_1^2}^{\text{bad!}}\rangle$$

This result is not part of the set, because

$$\underbrace{(cx_1)^2}_{\text{what the third component should be}} \neq \left\{cx_1^2\right\} \text{— what the third component is}$$

- If the operations for addition and multiplication are not changed, and the set passes axioms 1 and 6, you're done, because the set is a subspace of R^n (see 4.2).
- If the operations are changed, you will need to slog through all ten axioms until either:
  - All 10 axioms have been proven (the set is a vector space) or
  - You find an axiom that fails (the set is not a vector space).
- Refer to the next page for an example of each.

all vectors of this kind are in the set

- The set of all ordered pairs of real numbers, with the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$

\+ #1) $u + v = \underline{(u_1 + v_1, u_2 + v_2)}$ ✓

is an ordered pair of real #s

C #2) $u + v = v + u$

$\underbrace{(u_1 + v_1, u_2 + v_2) = (v_1 + u_1, v_2 + u_2)}_{\text{equal}}$ ✓

A #3) $(u + v) + w = u + (v + w)$ ✓

$(u_1 + v_1, u_2 + v_2) + (w_1, w_2) = (u_1, u_2) + (v_1 + w_1, v_2 + w_2)$

$(u_1 + v_1 + w_1, u_2 + v_2 + w_2) = (u_1 + v_1 + w_1, u_2 + v_2 + w_2)$

I #4) $u + \vec{0} = \vec{u}$ ✓

$(u_1 + u_2) + \underbrace{(0,0)}_{\text{zero vector}} = (u_1, u_2)$

N #5) $u + (-u) = \vec{0}$ ✓

$(u_1, u_2) + \underline{(-u_1, -u_2)} = (0, 0)$

is an ordered pair of real #s

$k\mathbf{u} = (0, ku_2)$

x #6) $ku = \underline{(0, ku_2)}$ ✓

is an ordered pair of real #s

D #7) $k(u + v) = ku + kv$

$k(u_1 + v_1, u_2 + v_2) = (0, ku_2) + (0, kv_2)$

$(0, k(u_2 + v_2)) = (0, k(u_2 + v_2))$ ✓

D #8) $(k + j)u = ku + ju$

$(0, (k + j)u_2) = (0, ku_2) + (0, ju_2)$

$(0, ku_2 + ju_2) = (0, ku_2 + ju_2)$ ✓

A #9) $k(ju) = (kj)u$ ✓

$k(0, ju_2) = (0, kju_2) = (0, kju_2)$

I #10) $1\vec{u} = \vec{u}$

$(0, 1u_2) \neq (u_1, u_2)$ ✗

Fails Axiom 10 – not a vector space.

- The set of all pairs of real numbers of the form $(1, x)$ with the operations:

★ let $\vec{u} = (1, x)$ and $\vec{v} = (1, y)$ and $\vec{w} = (1, z)$

$(1, y) + (1, y') = (1, y + y')$ $\qquad k(1, y) = (1, ky)$

\+ #1) $u + v = \underline{(1, x + y)}$ ✓

is an ordered pair of real #s $(1, x)$

C #2) $u + v = v + u$

$\underbrace{(1, x + y) = (1, y + x)}_{\text{equal}}$ ✓

A #3) $(u + v) + w = u + (v + w)$ ✓

$(1, x + y) + (1, z) = (1, x) + (1, y + z)$

$\underbrace{(1, x + y + z) = (1, x + y + z)}_{\text{equal}}$

I #4) $u + \vec{0} = \vec{u}$ ✓

$(1, x) + \underbrace{(1, 0)}_{\text{zero vector}} = (1, x)$

this set has a specific zero vector, because (0, 0) would not be allowed in the set. this vector (1, 0) can be added to any other vector in the set and give the same vector back, which is the requirement for Axiom 4.

N #5) $u + (-u) = \vec{0}$ ✓

$(1, x) + (1, -x) = (1, 0)$

x #6) $ku = \underline{(1, kx)}$ ✓

is an ordered pair of real #s $(1, x)$

D #7) $k(u + v) = ku + kv$ ✓

$k(1, x + y) = (1, kx) + (1, ky)$

$(1, k(x + y)) = (1, k(x + y))$

D #8) $(k + j)u = ku + ju$ ✓

$(1, (k + j)x) = (1, kx) + (1, jx)$

$(1, kx + jx) = (1, kx + jx)$

A #9) $k(ju) = (kj)u$ ✓

$k(1, jx) = (1, kjx) = (1, kjx)$

I #10) $1u = u$ ✓

$(1, 1x) = (1, x)$

Passes all 10 Axioms.
The set is a vector space.

4.2 – Subspaces

- To show if one set is a subspace of another vector space, you only need to check axioms 1 and 6. If both of them hold, you have a subspace.

- Checking if a vector is a linear combination of other vectors:

$\vec{u} = \langle 0,-2,2 \rangle$ $\vec{v} = \langle 1,3,-1 \rangle$

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is $(3,1,5)$ a linear combination of u and v?

$$\begin{array}{r} \text{first comp.}\rightarrow \\ \text{second}\rightarrow \\ \text{third}\rightarrow \\ \end{array} \begin{array}{cc|c} 0 & 1 & 3 \\ -2 & 3 & 1 \\ 2 & -1 & 5 \\ \uparrow u & \uparrow v & \uparrow \text{check} \end{array} \xrightarrow{\text{RREF}} \underbrace{\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]}_{\text{Because the matrix is consistent, it is a linear combination.}}$$

$$\begin{array}{r} 4u \\ +\,3v \\ \hline \langle 3,1,5 \rangle \end{array} \qquad \begin{array}{r} \langle 0,-8,8 \rangle \\ +\,\langle 3,9,-3 \rangle \\ \hline \langle 3,1,5 \rangle \end{array}$$

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is $(2,0,4)$ a linear combination of $\{(1,3,5),(5,-1,-7),(-3,-1,1)\}$?

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & 2 \\ 3 & -1 & -1 & 0 \\ 5 & -7 & 1 & 4 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ \hline \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} \end{array}\right] \text{— INCONSISTENT}$$

Because the matrix is inconsistent, it is not a linear combination.

- Checking if a matrix is a linear combination of other matrices:

is $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ a linear combo. of $\left\{ \underset{A}{\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}}, \underset{B}{\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}}, \underset{C}{\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}} \right\}$?

$$\text{position (row,col)} \left\{ \begin{array}{r} (1,1) \\ (1,2) \\ (2,1) \\ (2,2) \\ \end{array} \right. \begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \\ \uparrow A & \uparrow B & \uparrow C & \uparrow \text{check} \end{array} \xrightarrow{\text{RREF}} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{Consistent} \Rightarrow \text{is a L.C.}} \quad \begin{array}{r} 1A \\ +2B \\ +-3C \\ \hline \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} \end{array}$$

From our answer: $1\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + (-3)\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$

- You need at least n vectors to span n-space (R^n).
- Checking if n vectors span R^n:

$\vec{u} = \langle 3,1,4 \rangle$

$\vec{v} = \langle 2,-3,5 \rangle$

$\vec{w} = \langle 5,-2,9 \rangle$

$$\begin{array}{c} \begin{vmatrix} 3 & 2 & 5 \\ 1 & -3 & -2 \\ 4 & 5 & 9 \end{vmatrix} = 0 \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ u & v & w \end{array} \end{array}$$

X

The determinant is 0, so u, v, and w do not span $\mathbb{R}^3$.

$\vec{a} = \langle -1, 2, 0, 1 \rangle$

$\vec{b} = \langle 10, 4, 1, 8 \rangle$

$\vec{c} = \langle 2, -1, 2, -3 \rangle$

$\vec{d} = \langle 8, 0, 8, 7 \rangle$

$$\begin{array}{c} \begin{vmatrix} -1 & 10 & 2 & 8 \\ 2 & 4 & -1 & 0 \\ 0 & 1 & 2 & 8 \\ 1 & 8 & -3 & 7 \end{vmatrix} = -707 \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ a & b & c & d \end{array} \end{array}$$

✓

The determinant is not 0, so a, b, c, and d do span $\mathbb{R}^4$.

- You need at least n + 1 polynomials to span p_n.
- p_n is the set of all polynomials of degree n. You can treat it exactly like R^{n+1}.
- $p(x) = 2x^3 - x^2 + 7$ → becomes a vector in 4-space: $\langle 2, -1, 0, 7 \rangle$.
- Convert your polynomials to vectors and check if they span R^{n+1}. If they do, the polynomials span p_n.

- M_{mn} is the set of all matrices of size m × n. You need at least m * n matrices to span M_{mn}.
- Checking whether m * n matrices span M_{mn}

$A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 9 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -9 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 5 & 4 \\ 2 & 9 & 6 \end{bmatrix}$

$D = \begin{bmatrix} 4 & -2 & 7 \\ 5 & -9 & 9 \end{bmatrix}$ $E = \begin{bmatrix} 5 & 1 & 0 \\ 6 & 9 & 3 \end{bmatrix}$ $F = \begin{bmatrix} 6 & 3 & 0 \\ 3 & 1 & 10 \end{bmatrix}$

$$\begin{array}{c} \text{position} \\ \text{(row, col)} \end{array} \left\{ \begin{array}{c} (1,1) \\ (1,2) \\ (1,3) \\ (2,1) \\ (2,2) \\ (2,3 \end{array} \right. \begin{array}{c} \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & -2 & 1 & 3 \\ -2 & 3 & 4 & 7 & 0 & 0 \\ 0 & 1 & 2 & 5 & 6 & 3 \\ 9 & -9 & 9 & -9 & 9 & 1 \\ 1 & 3 & 6 & 9 & 3 & 10 \end{vmatrix} = -1680 \\ \begin{array}{cccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ A & B & C & D & E & F \end{array} \end{array}$$

✓

The determinant is not 0, so matrices A-F span $M_{2,3}$

4.3 – Linear Independence

- Checking if vectors are linearly independent:

$\vec{u} = \langle 1, -3, 9 \rangle \quad \vec{v} = \langle 2, -2, 10 \rangle \quad \vec{w} = \langle 5, 1, 13 \rangle$

$$\begin{array}{ccc} \begin{bmatrix} 1 & 2 & 5 \\ -3 & -2 & 1 \\ 9 & 10 & 13 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ u & v & w \end{array} \end{array} \xrightarrow{RREF} \underbrace{\begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}} \quad ✗$$

did not give the identity, so the set is linearly dependent.

$\vec{u} = \langle 2, -4 \rangle \quad \vec{v} = \langle 1, 3 \rangle$

$$\begin{array}{cc} \begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ u & v \end{array} \end{array} \xrightarrow{RREF} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \quad ✓$$

identity → the set is linearly independent.

If your matrix is not square, look to see if there is an identity in the top square. For example:

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \; ✗ \text{ L.D.} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \; ✓ \text{ L.I.}$$

- Shortcuts:
  - If you have more vectors than components in each vector, the set is linearly dependent.
  - If any vectors are scalar multiples of each other, the set is linearly dependent.

- The Wronskian:

Are $1, x^2, e^x$ linearly independent?

$$\begin{array}{c|ccc|cc} f(x) & 1 & x^2 & e^x & 1 & x^2 \\ f'(x) & 0 & 2x & e^x & 0 & 2x \\ f''(x) & 0 & 2 & e^x & 0 & 2 \end{array}$$

$(1)(2x)(e^x) - (1)(e^x)(2)$

$= 2e^x(x-1)$ ✓ L.I.

↑

Because the determinant does not equal 0, the set is linearly independent.

☆ If it ends up equaling 0, the set may or may not be linearly independent. You can not say for certain.

4.4 – Coordinates and Basis

- A basis for R^n requires exactly n vectors.
- Standard basis:

$$\mathbb{R}^4: \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$$

$$P_3: \{1, x, x^2, x^3\} \qquad P_2: \{1, x, x^2\}$$

$$M_2: \left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\} \qquad M_{1,3}: \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

- Checking if 3 vectors form a basis for R^3:
  - The vectors must span $\mathbb{R}^3$.
  - The vectors must be linearly independent.

Does $\{(1,6,4),(2,4,-1),(-1,2,5)\}$ form a basis for $\mathbb{R}^3$?

check span:

$$\begin{vmatrix}1&2&-1\\6&4&2\\4&-1&5\end{vmatrix} = 0 \;\times$$

Determinant $0 \to$ does not span

Not a basis for $\mathbb{R}^3$.

- Normally, vectors are written as a sum of standard basis vectors. For example, in R^n:

$$\mathbf{a}(1, 0, 0) + \mathbf{b}(0, 1, 0) + \mathbf{c}(0, 0, 1) = (\mathbf{a}, \mathbf{b}, \mathbf{c}).$$

- To find the coordinate vector of $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ relative to another basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$:

$$x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = (\mathbf{a}, \mathbf{b}, \mathbf{c})$$

Find c.v. of $\underset{\vec{a}}{(3,8,12)}$ relative to $\{\overset{\vec{v}_1}{(1,5,9)},\overset{\vec{v}_2}{(2,6,10)},\overset{\vec{v}_3}{(4,7,11)}\}$:

$$\begin{array}{cccc} \left[\begin{array}{ccc|c}1&2&4&3\\5&6&7&8\\9&10&11&12\end{array}\right] \\ \begin{array}{cccc}\uparrow&\uparrow&\uparrow&\uparrow\\ \vec{v}_1&\vec{v}_2&\vec{v}_3&\vec{a}\end{array}\end{array} \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c}1&0&0&-3\\0&1&0&5\\0&0&1&-1\end{array}\right] \quad \boxed{\langle -3, 5, -1\rangle}$$

$\uparrow$ (3,8,12) relative to the new basis

- To find the vector represented by $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ in another basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$:

Find vector represented by $(2,-3,-2)$ in basis from above:

$$2\langle 1,5,9\rangle + (-3)\langle 2,6,10\rangle + (-2)\langle 4,7,11\rangle = \boxed{\langle -12,-22,-34\rangle}$$

- Coordinate vector of a matrix relative to a basis of other matrices:

Find c.v. of $\begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$ relative to $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \boxed{\langle 1, 2, 3, 4 \rangle}$$

4.5 – Dimension

- The dimension of a system's solution space is equal to the number of parameters in the solution.
- Example system:

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$
$$2x_1 + 6x_2 \quad - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$
$$5x_3 + 10x_4 + 15x_6 = 0$$
$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$$

- To find basis and dimension:

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

6 variables
- 3 equations (in RREF)

3 parameters ☆ DIMENSION: 3

Find a parameterized solution:

$x_2 = r$ $\quad x_1 = -3r - 4s - 2t$

$x_4 = s$ $\quad x_3 = -2s$

$x_5 = t$ $\quad x_6 = 0$

$$\begin{array}{c|c} & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ r: & \langle -3, 1, 0, 0, 0, 0 \rangle \\ s: & \langle -4, 0, -2, 1, 0, 0 \rangle \\ t: & \langle -2, 0, 0, 0, 1, 0 \rangle \end{array} \longleftarrow \text{BASIS} ☆$$

4.6 – Change of Basis

- To find a transition matrix $\mathbf{P}_{A\to B}$

$$\left[\underset{\text{new basis}}{B}\;\middle|\;\underset{\text{old basis}}{A}\right] \xrightarrow{\text{RREF}} \left[\underset{\text{identity matrix}}{I}\;\middle|\;\underset{\text{transition matrix from basis A to B}}{P_{A\to B}}\right]$$

A = standard basis for $\mathbb{R}^3$

$B = \{(-3,0,3),(-3,2,-1),(1,6,-1)\}$

$C = \{(-6,-6,0),(-2,-6,4),(-2,-3,7)\}$

Find $P_{A\to B}$:

$$\left[\underbrace{\begin{matrix}-3 & -3 & 1\\ 0 & 2 & 6\\ 3 & -1 & -1\end{matrix}}_{B}\;\middle|\;\underbrace{\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}}_{A}\right] \xrightarrow{\text{RREF}} \left[\underbrace{\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}}_{I}\;\middle|\;\underbrace{\begin{matrix}-\frac{1}{18} & \frac{1}{18} & \frac{5}{18}\\ -\frac{1}{4} & 0 & -\frac{1}{4}\\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12}\end{matrix}}_{P_{A\to B}}\right]$$

Find $P_{B\to C}$

$$C\left\{\left[\begin{matrix}-6 & -2 & -2\\ -6 & -6 & -3\\ 0 & 4 & 7\end{matrix}\;\middle|\;\begin{matrix}-3 & -3 & 1\\ 0 & 2 & 6\\ 3 & -1 & -1\end{matrix}\right]\right\}B$$

$$\downarrow$$

$$I\left\{\left[\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\;\middle|\;\begin{matrix}\frac{1}{2} & \frac{3}{4} & \frac{1}{12}\\ -1 & -\frac{17}{12} & -\frac{17}{12}\\ 1 & \frac{2}{3} & \frac{2}{3}\end{matrix}\right]\right\}P_{B\to C}$$

Find $P_{C\to A}$

$$\begin{matrix}A\\ I\end{matrix}\left\{\left[\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\;\middle|\;\begin{matrix}-6 & -2 & -2\\ -6 & -6 & -3\\ 0 & 4 & 7\end{matrix}\right]\right\}\begin{matrix}C\\ P_{C\to A}\end{matrix}$$

(Already in RREF)

- Using transition matrices to convert from one basis to another

$\vec{w} = (-5, 8, -5)$ in basis A.

to convert to basis B:

$$P_{A\to B}[\vec{w}]_A = \begin{bmatrix}-1/18 & 1/18 & 5/18\\ -1/4 & 0 & -1/4\\ 1/12 & 1/6 & 1/12\end{bmatrix}\begin{bmatrix}-5\\ 8\\ -5\end{bmatrix} = \underset{\vec{w}\text{ relative to basis }B}{[\vec{w}]_B} = \left(-\frac{2}{3}, \frac{5}{2}, \frac{1}{2}\right)$$

to convert from basis B to basis C:

$$P_{B\to C}[\vec{w}]_B = \begin{bmatrix}1/2 & 3/4 & 1/12\\ -1 & -17/12 & -17/12\\ 1 & 2/3 & 2/3\end{bmatrix}\begin{bmatrix}-2/3\\ 5/2\\ 1/2\end{bmatrix} = [\vec{w}]_C = \left(\frac{19}{12}, \frac{-43}{12}, \frac{4}{3}\right)$$

to convert from basis C to basis A:

$$P_{C\to A}[\vec{w}]_C = \begin{bmatrix}-6 & -2 & -2\\ -6 & -6 & -3\\ 0 & 4 & 7\end{bmatrix}\begin{bmatrix}19/12\\ -43/12\\ 4/3\end{bmatrix} = [\vec{w}]_A = (-5, 8, -5)$$

To transition from one basis X to another basis Y, multiply $P_{X\to Y}$ by your vector that is in basis X.

4.8 – Rank and Nullity

- To find nullity, set up your matrix with a column of zeroes on the right, and find dimension. Borrowing the system and matrix from Page 8:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{basis for column space}} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{basis for row space}$$

6 variables
3 equations
3 parameters ← NULLITY: 3

Parameterize and find a solution for the system. (Shown on page 8).

- The basis for the null space:

same as page 8

$$\begin{Bmatrix} \{\langle -3, 1, 0, 0, 0, 0\rangle, \\ \langle -4, 0, -2, 1, 0, 0\rangle, \\ \langle -2, 0, 0, 0, 1, 0\rangle\} \end{Bmatrix}$$

- The basis for the row space is made up of rows with leading 1's:

Leading 1's are circled in red above.

The basis for the row space: (r_1, r_2, r_3)

- The basis for the column space is made up of the columns containing leading 1's:

The basis for the column space: (c_1, c_3, c_6)

- If the number of items in the row and column spaces is the same, the rank of the matrix is equal to the number of items.

Row and column space both of size 3:

RANK: 3

- For a matrix with m columns, rank + nullity = m

$3 + 3 = 6$ ✓

4.9 – Matrix Transformations from R^n to R^m

- A transformation matrix is applied to a vector through multiplication.

Transform vector $(-3,4)$ with matrix $\begin{bmatrix}-2&1\\0&3\end{bmatrix}$:

$$\begin{bmatrix}-2&1\\0&3\end{bmatrix}\begin{bmatrix}-3\\4\end{bmatrix}=\boxed{\langle 10,12\rangle}$$

- You can apply a transformation to several vectors at one time (an "image"):

Apply transformation $\begin{bmatrix}-2&1\\0&3\end{bmatrix}$ to $A(-3,4)$ $B(-1,0)$ $C(2,-2)$ $D(0,8)$

$$\begin{bmatrix}-2&1\\0&3\end{bmatrix}\begin{array}{c}\begin{bmatrix}-3&-1&2&0\\4&0&-2&8\end{bmatrix}\\ \begin{array}{cccc}\uparrow&\uparrow&\uparrow&\uparrow\\A&B&C&D\end{array}\end{array}=\begin{array}{c}\begin{bmatrix}10&2&-6&8\\12&0&-6&24\end{bmatrix}\\ \begin{array}{cccc}\uparrow&\uparrow&\uparrow&\uparrow\\A'&B'&C'&D'\end{array}\end{array}$$

$A'=(10,12)$ $B'=(2,0)$ $C'=(-6,-6)$ $D'=(8,24)$

- A list of transformation matrices, and what they do:

$$\text{REFLECTIONS}\left\{\begin{array}{ll}
\begin{bmatrix}-1&0\\0&1\end{bmatrix} & \text{Reflect about y-axis.}\\
\begin{bmatrix}1&0\\0&-1\end{bmatrix} & \text{Reflect about x-axis}\\
\begin{bmatrix}0&1\\1&0\end{bmatrix} & \text{Reflect about line } y=x\\
\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix} & \text{Reflect about xy-plane}\\
\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix} & \text{Reflect about xz-plane}\\
\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix} & \text{Reflect about yz-plane}
\end{array}\right.$$

★Whichever variables stayed positive are the ones you reflected about.

- More transformation matrices, and what they do:

PROJECTIONS

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ Project on to y-axis

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Project on to x-axis

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Project on to xy-plane

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Project on to xz-plane

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Project on to yz-plane

★ Whichever variables stayed 1 are the ones you projected on to.

COUNTERCLOCKWISE ROTATIONS

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ Rotate by θ about origin

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ Rotate by θ about x-axis

★ Whichever Variable Stayed 1 is the axis you rotate about.

★ NOTE THE SIGNS

$\begin{bmatrix} \cos\theta & 0 & \underline{\sin\theta} \\ 0 & 1 & 0 \\ \underline{-\sin\theta} & 0 & \cos\theta \end{bmatrix}$ Rotate by θ about y-axis

$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Rotate by θ about z axis

★ All rotations are counterclockwise. To rotate clockwise, use the coterminal angle $(2\pi - \theta)$.

CONTRACTION/ DILATION

$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ Contract / Dilate by a factor of k.

★ If $k < 1$, say contract.
If $k > 1$, say dilate.

- Even more transformation matrices, and what they do:

EXPANSION IN ONE DIRECTION

$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Expansion in x-direction by a factor of k

$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Expansion in y-direction by a factor of k

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}$ Expansion in z direction by a factor of k

* The variable that changed to a number other than 1 is the direction you expanded in.

SHEAR

$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ Shear by a factor of k in the x-dir.

$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ Shear by a factor of k in the y-dir.

* The row corresponds to the shear direction.
First row: x-direction
Second row: y-direction

You can not expand by a negative factor k.
A matrix with a negative k is actually two transformations:

$$\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

A reflection over the y-axis
followed by
An expansion in x-dir by a factor of 3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

A reflection over the xy-plane
followed by
An expansion in y-dir by a factor of 7.

4.10/11 – Properties/Compositions of Transformations

- Expressing a matrix as a product of elementary matrices, and as a series of transformations:

Express $\begin{bmatrix} -4 & 1 \\ 6 & 5 \end{bmatrix}$ as a product of elementary matrices and describe the geometric effect as a series of transformations:

Reduction Operations		Opposites	Elementary Matrix	Geometric Effect
	$\begin{bmatrix} -4 & 1 \\ 6 & 5 \end{bmatrix}$			
$-\frac{1}{4}R_1$	$\begin{bmatrix} 1 & -1/4 \\ 6 & 5 \end{bmatrix}$	$-4R_1$ →	$\begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$	Reflection over y-axis Expand by 4 in x-dir
R_2-6R_1	$\begin{bmatrix} 1 & -1/4 \\ 0 & 13/2 \end{bmatrix}$	R_2+6R_1 →	$\begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$	Shear by 6 in y-dir
$\frac{2}{13}R_2$	$\begin{bmatrix} 1 & -1/4 \\ 0 & 1 \end{bmatrix}$	$\frac{13}{2}R_2$ →	$\begin{bmatrix} 1 & 0 \\ 0 & 13/2 \end{bmatrix}$	Expand by $\frac{13}{2}$ in y-dir
$R_1+\frac{1}{4}R_2$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$R_1-\frac{1}{4}R_2$ →	$\begin{bmatrix} 1 & -1/4 \\ 0 & 1 \end{bmatrix}$	Shear by $\frac{-1}{4}$ in x-dir

ORDER ↑

$$\begin{bmatrix} -4 & 1 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 13/2 \end{bmatrix}\begin{bmatrix} 1 & -1/4 \\ 0 & 1 \end{bmatrix}$$

★ Product of elementary matrices.

Geometric effect

1. A shear by -1/4 in the x-direction
2. An expansion by 13/2 in the y-direction
3. A shear by 6 in the y-direction
4. An expansion by 4 in the x-direction
5. A reflection over the y-axis

- Expressing a series of transformations as a single transformation matrix:

Find a single transformation matrix which performs the following set of operations in $\mathbb{R}^3$ in the specified order:

1. Dilate by a factor of 2.
2. Reflect over the xz-plane.
3. Rotate 60° counterclockwise about the y-axis
4. Project on to the xy-plane

Operation	Transformation Matrix
1	$A=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$
2	$B=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$
3	$C=\begin{bmatrix}\frac{1}{2}&0&\frac{\sqrt{3}}{2}\\0&1&0\\-\frac{\sqrt{3}}{2}&0&\frac{1}{2}\end{bmatrix}$
4	$D=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$

ORDER

Single Transformation Matrix

$$T=DCBA$$

$$\boxed{T=\begin{bmatrix}1&0&\sqrt{3}\\0&-2&0\\0&0&0\end{bmatrix}} \leftarrow \text{Answer}$$

★ Your calculator should handle the multiplication of DCBA, but make sure you have the matrices in order! The first operation performed should be furthest to the right, and so forth. If doing it by hand, follow:

$$T=D\cdot(C\cdot(B\cdot A))$$

by working out the innermost parenthesis first (PEMDAS).

Apply T to the vector $\vec{\omega}=\langle -3,9,8\rangle$

$$T\vec{\omega}=\begin{bmatrix}1&0&\sqrt{3}\\0&-2&0\\0&0&0\end{bmatrix}\begin{bmatrix}-3\\9\\8\end{bmatrix}=\langle -3+8\sqrt{3},-18,0\rangle$$