Linear Algebra Name Jacob Magnuson Date 2016-09-10 Period 7
Exam #1 (Chapters 1 & 2)
SPRING 2012

USE THE METHODS SPECIFIED IN EACH PROBLEM, AND SHOW ALL STEPS TO RECEIVE FULL CREDIT!!

1. In the blanks next to each system, classify each system as consistent or inconsistent, and state whether there exists no solution, one solution, or an infinite number of solutions:

a. $6x - 2y = 3$
$-18x + 6y = -9$ consistent inf. # sol'ns

b. $6x - 2y = 3$
$3x + y = -1$ consistent one sol'n

c. $6x - 2y = 3$
$12x - 4y = -1$ inconsistent no sol'n

a) $\begin{bmatrix} 6 & -2 & 3 \\ -18 & 6 & -9 \end{bmatrix} \xrightarrow{-3R_1} \begin{bmatrix} -18 & 6 & -9 \\ -18 & 6 & -9 \end{bmatrix}$ 1 eqn. 2 vars. ∞ # sol'ns, consistent

b) $\begin{bmatrix} 6 & -2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$ consistent, 1 sol'n

c) $\begin{bmatrix} 6 & -2 & 3 \\ 12 & -4 & -1 \end{bmatrix} \xrightarrow{2R_1} \begin{bmatrix} 12 & -4 & 6 \\ 12 & -4 & -1 \end{bmatrix}$ inconsistent, no sol'n

2. Evaluate the determinant (row reduction or inspection):

a. $\begin{vmatrix} 2 & 0 & 0 \\ 43 & 14 & 0 \\ \sqrt{43} & 43 & -1 \end{vmatrix} = (2)(14)(-1) = \boxed{-28}$

b. $\begin{vmatrix} 1 & 8 & 6 \\ -3 & -24 & -18 \\ 5 & 5 & 4 \end{vmatrix} \xrightarrow{R_2+3R_1} \begin{vmatrix} 1 & 8 & 6 \\ 0 & 0 & 0 \\ 5 & 5 & 4 \end{vmatrix} = \boxed{0}$ (row of zeroes)

c. $\begin{vmatrix} 0 & 1 & 3 & 4 \\ 2 & -7 & 2 & 4 \\ -3 & 11 & 4 & 0 \\ 1 & -3 & 2 & 2 \end{vmatrix}$ $R_1 \leftrightarrow R_4$ (-1)

$\begin{vmatrix} 1 & -3 & 2 & 2 \\ 2 & -7 & 2 & 4 \\ -3 & 11 & 4 & 0 \\ 0 & 1 & 3 & 4 \end{vmatrix}$ $R_2 - 2R_1$ ——, $R_3 + 3R_1$ ——

$\begin{vmatrix} 1 & -3 & 2 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & 10 & 6 \\ 0 & 1 & 3 & 4 \end{vmatrix}$ $R_3 + 2R_2$ ——, $R_4 + R_2$ ——

$\begin{vmatrix} 1 & -3 & 2 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 1 & 4 \end{vmatrix}$ $R_4 - \frac{1}{6}R_3$ ——

$\begin{vmatrix} 1 & -3 & 2 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 3 \end{vmatrix} = (1)(-1)(6)(3) = -18(-1) = \boxed{18}$

3. Given the following matrices A, B, & C:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 5 \\ 1 & 6 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 0 & 1 \end{bmatrix}; \quad C = \begin{bmatrix} 2 & 6 \\ 1 & -1 \\ 5 & 0 \end{bmatrix}$$

a. Find CA

$$\begin{matrix} C & & A \\ 3 \times 2 & - & 3 \times 3 \\ & \neq & \end{matrix}$$

$$\boxed{\text{DNE}}$$

b. Find $A^2 - 2CB$

$$A^2 = \begin{bmatrix} 2 & 6 & 11 \\ 5 & 31 & -5 \\ 1 & -4 & 31 \end{bmatrix} - \underbrace{\begin{bmatrix} 52 & 12 & 4 \\ -6 & 6 & -6 \\ 10 & 30 & -20 \end{bmatrix}}_{2CB} = \boxed{\begin{bmatrix} -50 & -6 & 7 \\ 11 & 25 & 1 \\ -9 & -34 & 51 \end{bmatrix}}$$

$$\underset{\underbrace{3 \times 2 - 2 \times 3}}{CB} = \begin{bmatrix} 26 & 6 & 2 \\ -3 & 3 & -3 \\ 5 & 15 & -10 \end{bmatrix}$$

c. Find $(AC + 3B^T)^T$

$$\underset{\underbrace{3 \times 3 - 3 \times 2}}{AC} = \begin{bmatrix} 9 & 4 \\ 24 & 1 \\ 8 & 0 \end{bmatrix} \qquad 3B^T = \begin{bmatrix} 3 & 12 \\ 9 & 0 \\ -6 & 3 \end{bmatrix}$$

$$(AC + 3B^T)^T = \boxed{\begin{bmatrix} 12 & 33 & 2 \\ 16 & 1 & 3 \end{bmatrix}}$$

4. Find the quadratic polynomial whose graph passes through the points (-1,5), (0,4), and (1,9) using matrices.

$p(x) = ax^2 + bx^1 + cx^0$

$$\begin{array}{c|c} & \begin{array}{cccc} x^0 & x^1 & x^2 & p(x) \end{array} \\ \begin{array}{c} (-1,5) \\ (0,4) \\ (1,9) \end{array} & \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 1 & 0 & 0 & 4 \\ 1 & 1 & 1 & 9 \end{array}\right] \\ & \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ c & b & a \end{array} \end{array} \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 1 & -1 & 1 & 5 \\ 1 & 1 & 1 & 9 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 5 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 5 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{cccc} 1 & 0 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 6 \end{array}\right]$$

$\Downarrow$

$c = 4$

$b - a = -1$

$2a = 6$

$a = 3$

$b = 2$

$$\boxed{p(x) = 3x^2 + 2x + 4}$$

5. Write the system of equations as an augmented matrix, and solve the system by Gauss-Jordan elimination:

$$x_1 - x_2 - 5x_3 = -1$$
$$-2x_1 + 2x_2 + 11x_3 = 1$$
$$3x_1 - x_2 + x_3 = 3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -5 & -1 \\ -2 & 2 & 11 & 1 \\ 3 & -1 & 1 & 3 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & -5 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & 16 & 6 \end{array}\right] \tfrac{1}{2}R_3,\ R_2 \leftrightarrow R_3 \left[\begin{array}{ccc|c} 1 & -1 & -5 & -1 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$x_3 = -1$$
$$x_2 + 8(-1) = 3$$
$$x_2 = 11$$
$$x_1 - 1(11) - 5(-1) = -1$$
$$x_1 = 5$$

$$\boxed{(5, 11, -1)}$$

6. Evaluate the determinant using the cofactor method:

$$A = \begin{vmatrix} 2 & 2 & 3 & 0 & 5 \\ 3 & 3 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{vmatrix}$$

Using column $j=5$: $5(-1)^{1+5} \begin{vmatrix} 3 & 3 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{vmatrix} = (5)(-3) = \boxed{-15}$

Using column $j=4$: $1(-1)^{1+4} \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{vmatrix} = (-1)(3) = -3$

$(8+0+1)-(0+2+4) = 3$

7. Solve the system of equations using Cramer's Rule:

$$4x - z = 5$$
$$3y + z = 1$$
$$x - y = 2$$

$$A = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix} \qquad \det(A) = (0+0+0) - (-3-4) = 7$$

$$A_1 = \begin{bmatrix} 5 & 0 & -1 \\ 1 & 3 & 1 \\ 2 & -1 & 0 \end{bmatrix} \qquad \det(A_1) = (0+0+1) - (-6-5+0) = 12$$

$$A_2 = \begin{bmatrix} 4 & 5 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \qquad \det(A_2) = (0+5+0) - (-1+0+8) = -2$$

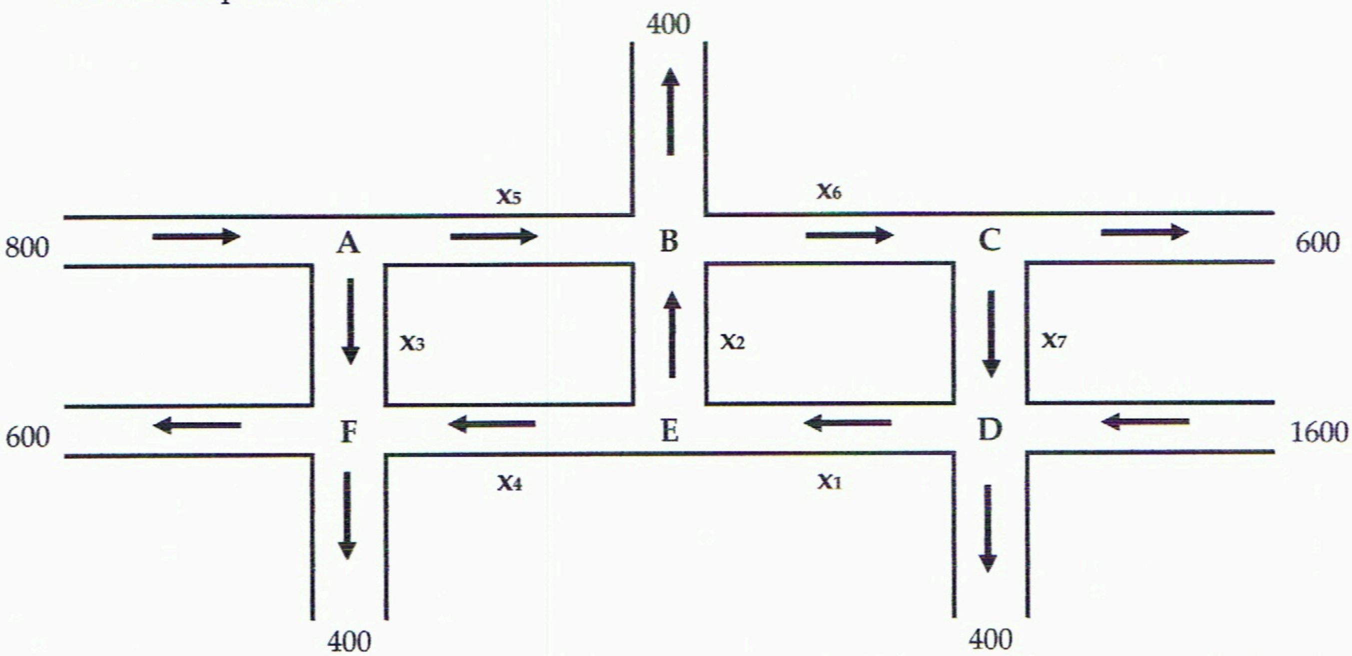
$$A_3 = \begin{bmatrix} 4 & 0 & 5 \\ 0 & 3 & 1 \\ 1 & -1 & 2 \end{bmatrix} \qquad \det(A_3) = (24+0+0) - (15-4+0) = 13$$

$$\frac{\det(A_1)}{\det(A)} = \boxed{x = \frac{12}{7}}$$

$$\frac{\det(A_2)}{\det(A)} = \boxed{y = \frac{-2}{7}}$$

$$\frac{\det(A_3)}{\det(A)} = \boxed{z = \frac{13}{7}}$$

8. The accompanying figure shows a network of one-way streets with traffic flowing in the directions indicated. The flow rates along the streets are measured as the average number of vehicles per hour.



a. Set up a linear system whose solution provides the unknown flow rates, and solve the system for the unknown flow rates (parameters will be utilized). (You must a linear system; matrices are optional.)

	in	=	out
A	800	=	$x_3 + x_5$
B	$x_2 + x_5$	=	$x_6 + 400$
C	x_6	=	$x_7 + 600$
D	$1600 + x_7$	=	$x_1 + 400$
E	x_1	=	$x_2 + x_4$
F	$x_3 + x_4$	=	1000

$$
\begin{array}{c}
\\ A \\ B \\ C \\ D \\ E \\ F
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\
\left[\begin{array}{ccccccc|c}
0 & 0 & 1 & 0 & 1 & 0 & 0 & 800 \\
0 & 1 & 0 & 0 & 1 & -1 & 0 & 400 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 600 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 1200 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1000
\end{array}\right]
\end{array}
$$

RREF

$$
\left[\begin{array}{ccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & -1 & 1200 \\
0 & 1 & 0 & 0 & 1 & 0 & -1 & 1000 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 800 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 200 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 600 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$x_1 = 1200 + s$

$x_2 = 1000 + s - t$

$x_3 = 800 - t$

$x_4 = 200 + t$

$x_5 = t$

$x_6 = 600 + s$

$x_7 = s$

b. Is it possible for the road from A to B and from B to C to be closed for construction (with traffic still flowing in their normal directions on the other streets?) Explain.

$A \rightarrow B: x_5$

$B \rightarrow C: x_6$

No. With $x_6 = 0$, s must be -600 (because $x_6 = 600 + s$).
This would correspond to a negative flow rate on x_7, which is illegal.

c. If the flow along the road from C to D must be reduced for construction, what is the minimum flow on each road that is required to keep traffic flowing?

$C \rightarrow D: x_7$

Letting $s = t = 0$:

$x_1 = 1200$

$x_2 = 1000$

$x_3 = 800$

$x_4 = 200$

$x_5 = 0$

$x_6 = 600$

$x_7 = 0$

9. Solve the system using an inverse matrix:

$$x_1 - 2x_2 - 6x_3 - 2x_4 = 4$$
$$x_1 + x_2 + 2x_3 + x_4 = -2$$
$$x_1 - x_3 = 1$$
$$3x_1 + 3x_2 + 7x_3 + 4x_4 = 3$$

$$\underbrace{\begin{bmatrix} 1 & -2 & -6 & -2 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 0 \\ 3 & 3 & 7 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 4 \\ -2 \\ 1 \\ 3 \end{bmatrix}}_{b}$$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 & 2 & -2 & 0 \\ -2 & 0 & 5 & -1 \\ 1 & 2 & -3 & 0 \\ -1 & -5 & 3 & 1 \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 4 \\ -2 \\ 1 \\ 3 \end{bmatrix}}_{b}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ -3 \\ 12 \end{bmatrix}$$