Linear Algebra
Exam #1 (Chapter 1)
Part 1 – GDC Calculator

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Date: 2016-09-11

Period: 7

1. Solve the system using an inverse matrix:

$$4.5x_1 - 7.8x_2 - 3.4x_3 + 2.1x_4 = 9.7$$
$$6.7x_1 - 4.5x_2 + 1.2x_3 - 5.3x_4 = 2.1$$
$$-3.9x_1 + 6.5x_2 - 11.9x_3 + 5.0x_4 = 8.7$$
$$-9.3x_1 + 10.9x_2 - 3.2x_3 + 3.1x_4 = -2.8$$

Note: you may use your GDC to solve this – your work should include the inverse and show the steps used to solve the system.

$$\begin{bmatrix} 4.5 & -7.8 & -3.4 & 2.1 \\ 6.7 & -4.5 & 1.2 & -5.3 \\ -3.9 & 6.5 & -11.9 & 5.0 \\ -9.3 & 10.9 & -3.2 & 3.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9.7 \\ 2.1 \\ 8.7 \\ -2.8 \end{bmatrix}$$

$$A \qquad x = b$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = (A^{-1}) \begin{bmatrix} 9.7 \\ 2.1 \\ 8.7 \\ -2.8 \end{bmatrix}$$

$$x = A^{-1} b$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1.735 \\ -1.854 \\ -1.770 \\ -1.416 \end{bmatrix}$$

2. Balance the following chemical reaction by solving a linear system:

$$Pb(N_3)_2 + Cr(MnO_4)_2 \rightarrow Cr_2O_3 + MnO_2 + Pb_3O_4 + NO$$

Note: you may use your GDC to solve this, but **you must use linear systems** in order to receive credit for your answer. Your work should include the linear system, the matrix you input to the calculator and the reduced row-echelon form of the matrix.

$$x_1(Pb(N_3)_2) + x_2(Cr(MnO_4)_2) = x_3(Cr_2O_3) + x_4(MnO_2) + x_5(Pb_3O_4) + x_6(NO)$$

Pb: $x_1 = 3x_5$

N: $6x_1 = x_6$

Cr: $x_2 = 2x_3$

Mn: $2x_2 = x_4$

O: $8x_2 = 3x_3 + 2x_4 + 4x_5 + x_6$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -3 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 8 & -3 & -2 & -4 & -1 & 0 \end{array}\right]$$

RREF

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -1/6 & 0 \\ 0 & 1 & 0 & 0 & 0 & -22/45 & 0 \\ 0 & 0 & 1 & 0 & 0 & -11/45 & 0 \\ 0 & 0 & 0 & 1 & 0 & -44/45 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1/18 & 0 \end{array}\right]$$

$x_1 = \frac{1}{6}t$

$x_2 = \frac{22}{45}t$

$x_3 = \frac{11}{45}t$

$x_4 = \frac{44}{45}t$

$x_5 = \frac{1}{18}t$

$x_6 = t$

$\text{lcm}(6, 45, 18) = 90$

$t = 90k,\ k \in \mathbb{N}$

i.e. $15\,Pb(N_3)_2 + 44\,Cr(MnO_4)_2 \rightarrow 22\,Cr_2O_3 + 88\,MnO_2 + 5\,Pb_3O_4 + 90\,NO$

Linear Algebra
Exam #1 (Chapters 1 & 2)
Part 2 – No Calculator
SPRING 2013

Name: Jacob Magnuson   Date: 2016-09-11   Period: 7

USE THE METHODS SPECIFIED IN EACH PROBLEM, AND SHOW ALL STEPS TO RECEIVE FULL CREDIT!!

1. In the blanks next to each system, classify each system as consistent or inconsistent, and state whether there exists no solution, one solution, or an infinite number of solutions:

a. $$\begin{aligned} x_1 + 2x_2 + x_3 &= 7 \\ x_1 - x_3 &= 3 \\ 4x_1 + x_2 + x_3 &= 10 \end{aligned}$$ consistent ____ one sol'n ____

b. $$\begin{aligned} x_1 + 2x_3 &= 7 \\ x_2 - 6x_3 &= 8 \end{aligned}$$ consistent ____ inf. # of sol'ns ____

c. $$\begin{aligned} 3x_1 + 10x_2 &= 7 \\ 9x_1 &= -30x_2 + 35 \end{aligned}$$ inconsistent ____ no sol'n ____

a) $$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 0 & -1 & 3 \\ 4 & 1 & 1 & 10 \end{array}\right] \xrightarrow[R_2-R_1,\ R_3-4R_1]{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 5 & -2 \end{array}\right] \xrightarrow[R_3-R_2]{\frac{1}{2}R_2,} \left[\begin{array}{cccc} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 4 & -4 \end{array}\right]$$ consistent one sol'n

b) 2 eqn's, 3 vars   consistent, inf # sol'ns

c) $$\left[\begin{array}{cc|c} 3 & 10 & 7 \\ 9 & 30 & 35 \end{array}\right] \xrightarrow[\frac{1}{3}R_2]{} \left[\begin{array}{cc|c} 3 & 10 & 7 \\ 3 & 10 & \frac{35}{3} \end{array}\right]$$ inconsistent, no sol'n

2. Given the following matrices A, B, & C:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 4 \end{bmatrix}; \quad D = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}; \quad E = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}$$

a. Find CE

$$\underset{3 \times 2 \quad 2 \times 2}{C \; E} = \begin{bmatrix} 4 & 6 \\ 5 & 7 \\ 12 & 20 \end{bmatrix}$$

b. Find CB

$$\underset{3 \times 2 \;//\; 3 \times 3}{C \qquad B}$$

$\boxed{\text{DNE}}$

c. Find $(3B^T)^T$

$$(3B^T)^T = 3B = \begin{bmatrix} -3 & 3 & 0 \\ 0 & 3 & 3 \\ 3 & -3 & -3 \end{bmatrix}$$

d. Write a matrix **F** that would be added to A to transform A into an upper triangular matrix.

$$F = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. The following shows the solution of x and y for a system of equations using Cramer's Rule. From this information, set up the solution for z, and state the value for z?

$$x = \frac{\begin{vmatrix} 1 & -3 & -1 \\ 3 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}}$$

$$A = \begin{bmatrix} 1 & -3 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$y = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}}$$

$$z = \frac{\det(A_3)}{\det(A)} = \frac{\begin{vmatrix} 1 & -3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}} = \frac{(0-9+4)-(1+0+6)}{(2-3-4)-(-1-12+2)} = \frac{-12}{6} = \boxed{-2}$$

4. Find the quadratic polynomial whose graph passes through the points (0,-1), (1,4), (-1,0), and (2,15) using matrices.

$p(x) = ax^2 + bx^1 + cx^0$ degree 2; only 3 pts. needed

$$\begin{array}{c|c}
 & \begin{matrix} x^0 & x^1 & x^2 & p(x) \end{matrix} \\
\begin{matrix} (0,-1) \\ (1,4) \\ (-1,0) \end{matrix} & \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 0 \end{array}\right] \\
 & \begin{matrix} c & b & a \end{matrix}
\end{array}
\xrightarrow[R_3-R_1]{R_2-R_1}
\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 5 \\ 0 & -1 & 1 & 1 \end{array}\right]
\xrightarrow{R_3+R_2}
\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 6 \end{array}\right]$$

$c = -1$

$b + a = 5$

$2a = 6$

$a = 3$

$b = 2$

$$\boxed{p(x) = 3x^2 + 2x - 1}$$

5. Write the system of equations as an augmented matrix, and solve the system by Gauss-Jordan elimination:

$$3x_1 - x_2 + 7x_3 = 0$$
$$2x_1 - x_2 + 4x_3 = \frac{1}{2}$$
$$x_1 - x_2 + x_3 = 1$$
$$6x_1 - 4x_2 + 10x_3 = 3$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 7 & 0 \\ 2 & -1 & 4 & \frac{1}{2} \\ 1 & -1 & 1 & 1 \\ 6 & -4 & 10 & 3 \end{array}\right] \xrightarrow[2R_2]{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 4 & -2 & 8 & 1 \\ 3 & -1 & 7 & 0 \\ 6 & -4 & 10 & 3 \end{array}\right] \xrightarrow[R_3 - 3R_1,\ R_4 - 6R_1]{R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 4 & -3 \\ 0 & 2 & 4 & -3 \\ 0 & 2 & 4 & -3 \end{array}\right]$$

$$\xrightarrow[R_3 - R_2,\ R_4 - R_2,]{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let $x_3 = t$

$x_2 + 2t = -\frac{3}{2}$

$x_2 = -\frac{3}{2} - 2t$

$x_1 - \left(-\frac{3}{2} - 2t\right) + t = 1$

$x_1 + 3t + \frac{3}{2} = 1$

$x_1 = -\frac{1}{2} - 3t$

$$\boxed{\left(-\frac{1}{2} - 3t,\ \frac{-3}{2} - 2t,\ t\right)}$$

6. A simplified economy has three interacting sectors: energy, food, and manufactured goods. It takes \$0.20 worth of energy (but no food or goods) to produce \$1 of energy; it takes \$0.10 of energy and \$0.10 of goods to produce \$1 of food; and it takes \$0.30 of energy and \$0.50 of goods to produce \$1 of goods. How many dollars should be produced by each sector in order to meet a demand for \$400 (worth) of energy, \$500 of food, and \$200 of goods?

$$\begin{array}{cc} & \text{Needs} \\ & \begin{array}{ccc} E & F & M \end{array} \\ \text{Producer} \begin{array}{c} E \\ F \\ M \end{array} & \begin{bmatrix} 0.20 & 0.10 & 0.30 \\ 0 & 0 & 0 \\ 0 & 0.10 & 0.50 \end{bmatrix} \end{array} = C \qquad I_3 - C = \begin{bmatrix} 0.80 & -0.10 & -0.30 \\ 0 & 1 & 0 \\ 0 & -0.10 & 0.50 \end{bmatrix}$$

invert $(I_3 - C)$:

$$\left[\begin{array}{ccc|ccc} .8 & -.1 & -.3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -.1 & .5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + \frac{1}{10}R_2} \left[\begin{array}{ccc|ccc} .8 & -.1 & -.3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & .5 & 0 & .1 & 1 \end{array}\right] \begin{array}{c} \frac{5}{4}R_1 \\ \\ 2R_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & -.125 & -.375 & 1.25 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & .2 & 2 \end{array}\right] \xrightarrow{R_1 + \frac{1}{8}R_2 + \frac{3}{8}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1.25 & .2 & .75 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & .2 & 2 \end{array}\right]$$

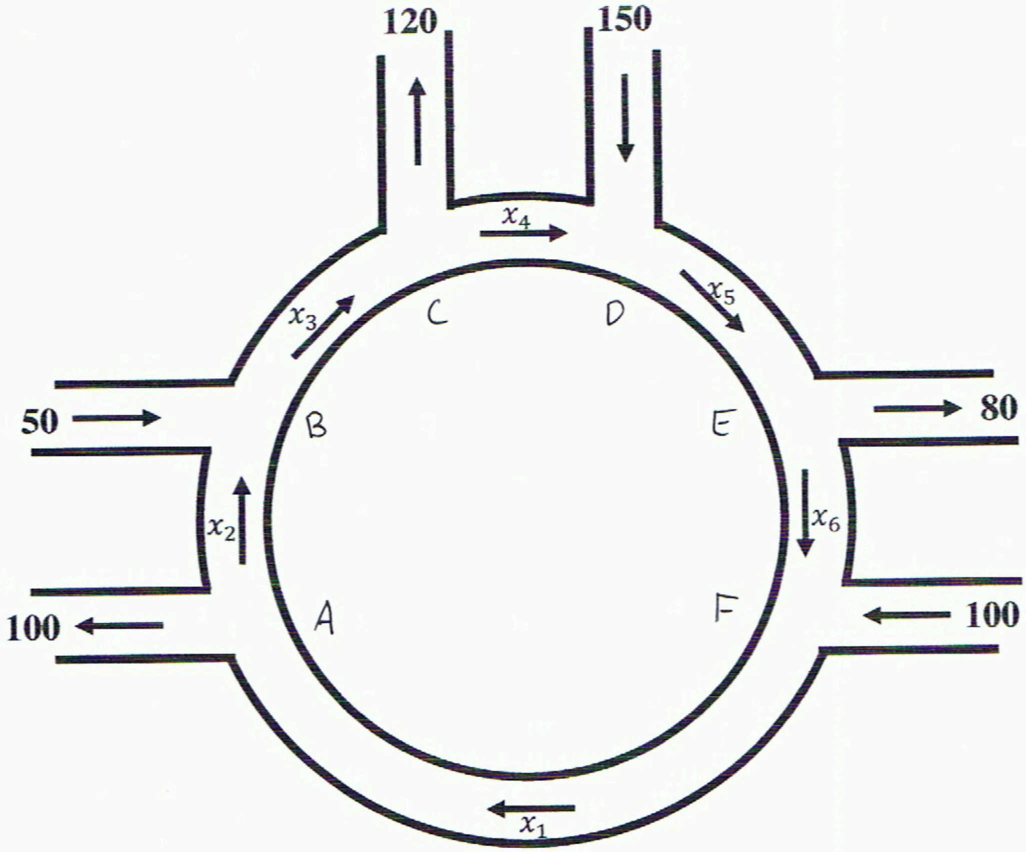
$\Uparrow$

$(I_3 - C)^{-1}$

production $\longrightarrow$ $x = (I_3 - C)^{-1} d$

$$\begin{bmatrix} 1.25 & .2 & .75 \\ 0 & 1 & 0 \\ 0 & .2 & 2 \end{bmatrix} \begin{bmatrix} 400 \\ 500 \\ 200 \end{bmatrix} = \begin{bmatrix} \$750 \\ \$500 \\ \$500 \end{bmatrix} \begin{array}{l} \text{energy} \\ \text{food} \\ \text{man. goods} \end{array}$$

7. The accompanying figure shows a typical intersection in England, constructed as a one-way "roundabout". Assume that traffic must travel in the directions shown. The flow rates along the streets are measured as the average number of vehicles per hour.



a. Use this space to set up a linear system whose solution provides the unknown flow rates.

(Note the intersection labels made above.)

in = out

A: $x_1 = x_2 + 100$

B: $x_2 + 50 = x_3$

C: $x_3 = x_4 + 120$

D: $x_4 + 150 = x_5$

E: $x_5 = x_6 + 80$

F: $x_6 + 100 = x_1$

b. Solve the system for the general solution of the network flow – parameters will be utilized. (You must use a linear system with matrices.)

$$\begin{array}{cc}
 & \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} &
\left[\begin{array}{cccccc|c}
1 & -1 & 0 & 0 & 0 & 0 & 100 \\
0 & 1 & -1 & 0 & 0 & 0 & -50 \\
0 & 0 & 1 & -1 & 0 & 0 & 120 \\
0 & 0 & 0 & 1 & -1 & 0 & -150 \\
0 & 0 & 0 & 0 & 1 & -1 & 80 \\
-1 & 0 & 0 & 0 & 0 & 1 & -100
\end{array}\right]
\end{array}
\xrightarrow{\text{RREF}}
\left[\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & 0 & -1 & 100 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 50 \\
0 & 0 & 0 & 1 & 0 & -1 & -70 \\
0 & 0 & 0 & 0 & 1 & -1 & 80 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

$$x_1 = 100 + t$$

$$x_2 = t$$

$$x_3 = 50 + t$$

$$x_4 = t - 70$$

$$x_5 = 80 + t$$

$$x_6 = t$$

c. Find the smallest possible value for x_6.

$$x_{6,min} = t_{min} = \boxed{70} \quad (\text{as } x_4 = t - 70)$$

8. Evaluate the following determinants using the designated method (your work should reflect that you used the indicated method):

a. $\begin{vmatrix} 6 & 3 & 9 & -12 \\ 5 & 0 & 1 & 2 \\ \frac{4}{5} & \frac{2}{5} & \frac{6}{5} & -\frac{8}{5} \\ \frac{2}{3} & -\frac{1}{3} & \frac{5}{3} & \frac{8}{3} \end{vmatrix}$; inspection

$R_1 = \frac{15}{2} R_3$, $\therefore$ det $= \boxed{0}$

b. $\begin{vmatrix} 0 & 1 & 3 & 4 \\ 2 & -7 & 2 & 4 \\ -3 & 11 & 4 & 0 \\ 1 & -3 & 2 & 2 \end{vmatrix}$; row-reduction

$\xrightarrow[(-1)]{R_1 \leftrightarrow R_4}$ $\begin{vmatrix} 1 & -3 & 2 & 2 \\ 2 & -7 & 2 & 4 \\ -3 & 11 & 4 & 0 \\ 0 & 1 & 3 & 4 \end{vmatrix}$ $\begin{matrix} \\ R_2 - 2R_1 \\ R_3 + 3R_1 \\ \\ \end{matrix}$

$\begin{vmatrix} 1 & -3 & 2 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & 10 & 6 \\ 0 & 1 & 3 & 4 \end{vmatrix}$ $\begin{matrix} \\ \\ R_3 + 2R_2 \\ R_4 + R_2 \end{matrix}$

$\begin{vmatrix} 1 & -3 & 2 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 1 & 4 \end{vmatrix}$ $\begin{matrix} \\ \\ \\ R_4 - \frac{1}{6} R_3 \end{matrix}$

$\begin{vmatrix} 1 & -3 & 2 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 3 \end{vmatrix} = (1)(-1)(6)(3) = (-18)(-1) = \boxed{18}$

c. $\begin{vmatrix} 2 & k & 3 & 0 \\ k & 4 & 1 & 1 \\ 0 & -1 & 5 & 1 \\ 0 & 1 & 4 & 2 \end{vmatrix}$; cofactor expansion

Using column $j = 1$:

$2(-1)^{1+1} \begin{vmatrix} 4 & 1 & 1 \\ -1 & 5 & 1 \\ 1 & 4 & 2 \end{vmatrix} + K(-1)^{2+1} \begin{vmatrix} K & 3 & 0 \\ -1 & 5 & 1 \\ 1 & 4 & 2 \end{vmatrix} = 2(18) - K(6K + 9)$

$= \boxed{-6k^2 - 9k + 36}$

$(40 + 1 - 4) - (5 - 2 + 16)$

18

$(10K + 3 + 0) - (0 - 6 + 4K)$

$6K + 9$