Linear Algebra
Exam #2 (Chapters 3 & 4)
FALL 2016

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SHOW ALL STEPS TO RECEIVE FULL CREDIT!! Graphing calculators are permitted.

1. Write objects (vectors, matrices, polynomials) to satisfy each of the following:

a. the zero vector of the vector space $\mathbb{R}^6$

$$\boxed{\langle 0,0,0,0,0,0 \rangle}$$

b. the zero vector of the vector space M_{23}

$$\boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

c. the standard basis for P_4

$$\boxed{S = \{1, x, x^2, x^3, x^4\}}$$

d. the standard basis for M_{33}

$$\boxed{S = \left\{ \begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\1&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\1&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&1\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix} \right\}}$$

1 2 3 4 5 6 7 8 9

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2. Use vectors $\mathbf{a} = \langle 2, -1, 2 \rangle$, $\mathbf{b} = \langle 4, -2, 1 \rangle$, $\mathbf{c} = \langle -1, 1, -2 \rangle$, & $\mathbf{d} = \langle 3, 0, -1 \rangle$

a. Find a unit vector $\mathbf{u}$ with the same direction as $\mathbf{b}$.

$\|\vec{b}\| = \sqrt{16+4+1} = \sqrt{21}$

$\boxed{\vec{u} = \left\langle \frac{4}{\sqrt{21}}, \frac{-2}{\sqrt{21}}, \frac{1}{\sqrt{21}} \right\rangle}$

b. Find the dot product of $\mathbf{b} \cdot \mathbf{c}$.

$\vec{b} \cdot \vec{c} = 4(-1) + (-2)(1) + (1)(-2) = \boxed{-8}$

c. Find the angle between $\mathbf{a}$ & $\mathbf{c}$.

$\langle \frac{\sqrt{3}}{4}, \frac{1}{4} \rangle$

$\langle 2, 2\sqrt{3} \rangle$

$\cos^{-1}\left(\frac{\vec{a} \cdot \vec{c}}{\|a\|\|c\|}\right) = \boxed{162.284^\circ}$

d. Are $\mathbf{a}$ & $\mathbf{b}$ parallel, orthogonal, or neither? Justify your answer.

$\vec{a} \cdot \vec{b} = 13$

$\|a\| = 3$

$\|b\| = \sqrt{21}$

$13 \neq 0$ ← orthogonal

and

$\frac{13}{3\sqrt{21}} \neq 1$ ↖ parallel

$\therefore$ $\boxed{\text{neither}}$

3. Use vectors $\mathbf{a} = \langle 2,-1,2\rangle$, $\mathbf{b} = \langle 4,-2,1\rangle$, $\mathbf{c} = \langle -1,1,-2\rangle$, & $\mathbf{d} = \langle 3,0,-1\rangle$

a. Find a unit vector **u** with the same direction as **b**.

b. Find the dot product of $\mathbf{b}\cdot\mathbf{c}$.

repeats

c. Find the angle between **a** & **c**.

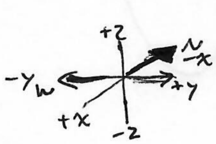
d. Find the vector component of **a** along **d**.

$$\left(\frac{\vec{a}\cdot\vec{d}}{\|d\|^2}\right)\vec{d} = \frac{4}{10}\langle 3,0,-1\rangle = \boxed{\left\langle \frac{6}{5}, 0, -\frac{2}{5}\right\rangle}$$

e. Are **a** & **b** parallel, orthogonal, or neither? Justify your answer.

repeated

f. Suppose vector **w** is pointing west, and vector **n** is pointing north. Which direction is the cross product pointing? Depends on $\vec{w}\times\vec{n}$ vs $\vec{n}\times\vec{w}$ ...



$$\left|\begin{array}{ccc|cc} i & j & k & i & j \\ 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 \end{array}\right| \quad 1k \quad \langle 0,0,1\rangle$$

$\vec{w}\times\vec{n} \Rightarrow \boxed{\text{UP}}$

$\langle 0,-1,0\rangle \times \langle -1,0,0\rangle$

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4. Determine whether the following subset is a subspaces of the given vector space. Justify your answer. The set $W = \{(x_1, 3, x_3)\colon\ x_1\ \&\ x_3$ are real numbers$\}$ – subspace of the vector space $\mathbb{R}^3$ (with the standard operations)?

$$(a,3,b)+(c,3,d)=\underbrace{(a+c,6,b+d)}_{\text{not in } W}$$

not closed under addition

$\therefore$ $\boxed{\text{not a vector space}}$

5. Determine if the set $S = \{\langle -2,3,0,0\rangle,\ \langle 3,0,-2,0\rangle, \langle 2,0,0,6\rangle, \langle 0,0,6,0\rangle\}$ is a basis for $\mathbb{R}^4$.

1) Span: $M=\begin{bmatrix} -2 & 3 & 2 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & 6 & 0 \end{bmatrix}$ $\det(M) = 324 \neq 0$

2) L.I.: $\operatorname{rref}(M) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ L.I.

$\boxed{\text{is a basis for } \mathbb{R}^4}$

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6. Let S be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$:

$$\mathbf{u} + \mathbf{v} = \langle 3v_1 + 3v_2, -u_1 - u_2 \rangle, \qquad k\mathbf{u} = \langle 3ku_2, -ku_1 \rangle$$

Determine if this set is a vector space. If not, find **three axioms** that fail.

$$1\langle u_1, u_2 \rangle = \langle 3u_2, -u_1 \rangle \neq \vec{u}$$

failed 1 $\boxed{\text{Fails mult. Identity}}$ ← Axiom 10

not a vector space

$$a(b\langle u_1, u_2 \rangle) = (ab)\langle u_1, u_2 \rangle$$

$$a\langle 3bu_2, -bu_1 \rangle = \langle +3abu_2, -abu_1 \rangle$$

$$\langle -3abu_1, -3abu_2 \rangle \neq$$

failed 2 $\boxed{\text{Fails Assoc. prop of scalar mult.}}$ ← Axiom 9

~~[illegible]~~

$\cancel{\langle u_1, u_2 \rangle}$

zero vector

$$\langle a, b \rangle + \langle \vec{0}_1, \vec{0}_2 \rangle = \langle a, b \rangle$$

$$a = 3\vec{0}_1 + 3\vec{0}_2$$

$$b = -a - b$$

$$2b = a \quad \times$$

Fails axiom 4

failed 3 $\boxed{\text{No zero vector}}$

0

7. Find the basis and dimension of the solution space of $A\mathbf{x} = \mathbf{0}$

$$A = \begin{bmatrix} 1 & 2 & -9 \\ 2 & 8 & -38 \\ 5 & 14 & -65 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\cancel{x_1 =}$

$\cancel{x_2 = s}$

$\cancel{x_3 = t}$

$x_1 = -t$

$x_2 = 5t$ $\quad t\langle -1, 5, 1\rangle$

$x_3 = t$

$$\boxed{\begin{array}{l} \text{basis: } S = \{\langle -1, 5, 1\rangle\} \\ \text{dim} = 1 \end{array}}$$

space: all of form $(x, -5x, -x)$

8. Determine a basis for the null space of the following matrix, and state the nullity:

with zeroes $\nearrow$ $A_2 = A = \begin{bmatrix} 3 & 5 & 37 & 12 & 0 \\ 1 & -5 & -21 & -4 & 0 \\ 2 & 15 & 83 & 22 & 0 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 2 & 0 \\ 0 & 1 & 5 & \frac{6}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$

$x_4 = t$

$x_1 = -4s - 2t$

$x_2 = -5s - \frac{6}{5}t$

$$\boxed{\begin{array}{l} s\langle -4, -5, 1, 0\rangle \\ + t\langle -2, -\frac{6}{5}, 0, 1\rangle \end{array}}$$

BASIS

$\boxed{\text{nullity: } 2}$

23/12

4.111

9. Find the transition matrix from $B = \{\langle 7,-3,3\rangle, \langle 0,-2,4\rangle, \langle 2,1,1\rangle\}$ to
$C = \{\langle 3,1,1\rangle, \langle 1,0,0\rangle, \langle 0,3,1\rangle\}$

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 7 & 0 & 2 \\ 1 & 0 & 3 & -3 & -2 & 1 \\ 1 & 0 & 1 & 3 & 4 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 7 & 1 \\ 0 & 1 & 0 & -11 & -21 & -1 \\ 0 & 0 & 1 & -3 & -3 & 0 \end{array}\right]$$

$\uparrow$
$P_{B\to C}$

$$\boxed{P_{B\to C} = \begin{bmatrix} 6 & 7 & 1 \\ -11 & -21 & -1 \\ -3 & -3 & 0 \end{bmatrix}}$$

$[x]_B = [3,-4,11]^T$ $\qquad$ $[x]_C = P_{B\to C}[x]_B = [1,40,3]^T$

$\begin{bmatrix} 3 \\ -4 \\ 11 \end{bmatrix}$ $\qquad$ $\langle 43, 10, 4\rangle$ $\qquad$ $\langle 43, 10, 4\rangle$

10. Find the coordinate matrix of $p(x) = x^2 - 15x + 2$ relative to the standard basis in P_2.

$B = \{1, x, x^2\}$

$\boxed{\cancel{[p(x)]_B = \langle 2, -15, 1\rangle}}$

$[p(x)]_B = \boxed{[2, -15, 1]^T}$

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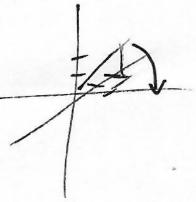
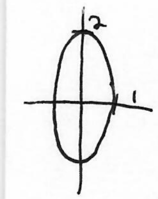
11. Find the image of the vector $\langle -2, 1, 2\rangle$:

a. Rotated 30° about the x-axis



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \boxed{\left\langle -2, \frac{\sqrt{3}}{2} - 1, \frac{1}{2} + \sqrt{3} \right\rangle}$$



b. Projected onto the yz-plane

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \boxed{\langle 0, 1, 2\rangle}$$

c. Reflected about the xz-plane

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \boxed{\langle -2, -1, 2\rangle}$$

?

12. Find the standard matrix for the following composition in $\mathbb{R}^2$.
A rotation (1) of 60°, followed by a projection (2) on the x-axis, followed by a reflection (3) about the line $y = x$, followed by a contraction (4) with a factor of $k = \frac{1}{2}$.

ccw?
cw?
assume ccw

$$\underset{4}{\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}}\underset{3}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\underset{2}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}\underset{1}{\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}} = \boxed{\begin{bmatrix} 0 & 0 \\ \frac{1}{4}, & -\frac{\sqrt{3}}{4} \end{bmatrix}}$$

$\cancel{(0,1)}$

$(1,0) \to \cancel{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \to \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(\frac{\sqrt{3}}{4}, \frac{1}{4}\right)}$

$\to \cancel{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}$

$\to \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \to \left(\frac{1}{2}, 0\right) \to \left(0, \frac{1}{2}\right) \to \left(0, \frac{1}{4}\right)$

13. Describe (in words) the geometric effect of multiplying a vector **x** by $A = \begin{bmatrix} 2 & 6 \\ 0 & -2 \end{bmatrix}$

$$\begin{bmatrix} 2 & 6 \\ 0 & -2 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

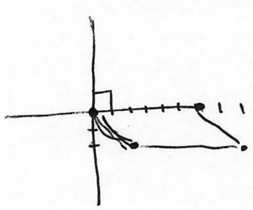
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1+3R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{-2R_2} \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \xrightarrow{3R_1} \begin{bmatrix} 2 & 6 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Split up

1) Shear by a factor of 3 in the x-direction
2) Reflect over x-axis
3) Expand by a factor of 2 in the y-direction
4) Expand by a factor of 3 in the x-direction

BONUS: The square with vertices (0,0), (1,0), (0,1), & (1,1) is transformed so that it has vertices at (0,0), (2,-2), (6,0), & (8,-2). What transformation matrix would be used to accomplish this?



1: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

2: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

3: $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

4: $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$\boxed{\begin{bmatrix} 6 & 2 \\ 0 & -2 \end{bmatrix}}$$

+5

$$\begin{bmatrix} 6 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 2 & 8 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

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