Linear Algebra
Exam #3 (Chapter 5)
SPRING 2012

Jacob Magnuson
Name

2015-11-20
Date

7
Period

USE THE METHODS SPECIFIED IN EACH PROBLEM, AND SHOW ALL STEPS TO RECEIVE FULL CREDIT!!

1. Find the characteristic polynomial of the following matrices:

a. $A = \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$ $\quad \lambda I - A = \begin{bmatrix} \lambda-2 & -5 \\ -4 & \lambda-1 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda-2)(\lambda-1) - (-5)(-4)$

$0 = \lambda^2 - 3\lambda + 2 - 20$

$\boxed{0 = \lambda^2 - 3\lambda - 18}$

OR

$0 = \lambda^2 - tr(A)\lambda + \det(A)$

$0 = \lambda^2 - 3\lambda + ((2)(1) - (4)(5))$

$\boxed{0 = \lambda^2 - 3\lambda - 18}$

b. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 4 \\ 6 & 4 & 5 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda-1 & -2 & -3 \\ -3 & \lambda & -4 \\ -6 & -4 & \lambda-5 \end{bmatrix} \begin{matrix} \lambda-1 & -2 \\ -3 & \lambda \\ -6 & -4 \end{matrix}$

$\det(\lambda I - A) = (\lambda-1)(\lambda)(\lambda-5) + (-2)(-4)(-6) + (-3)(-3)(-4) - (-3)(\lambda)(-6) - (\lambda-1)(-4)(-4) - (-2)(-3)(\lambda-5)$

$= \lambda^3 - 6\lambda^2 + 5\lambda - 48 - 36 - 18\lambda - 16\lambda + 16 - 6\lambda + 30$

$\boxed{0 = \lambda^3 - 6\lambda^2 - 35\lambda - 38}$

c. $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 3 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

eigenvalues

$\boxed{0 = (\lambda-1)(\lambda-3)(\lambda-5)(\lambda-6)}$

2. For the following matrix, $A = \begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix}$

a. Find all eigenvalues (classify multiplicity if needed).

$$0 = \lambda^2 + 3\lambda - 10$$

$$0 = (\lambda+5)(\lambda-2)$$

$$\boxed{\lambda_1 = -5 \qquad \lambda_2 = 2}$$

b. Find corresponding eigenvectors and bases.

$$\lambda I - A = \begin{bmatrix} \lambda-3 & 4 \\ -2 & \lambda+6 \end{bmatrix}$$

$$\lambda_1: \quad -5I - A = \begin{bmatrix} -8 & 4 \\ -2 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = \frac{1}{2}t \\ x_2 = t \end{matrix} \rightarrow v_1 = t\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \rightarrow \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$\lambda_2: \quad 2I - A = \begin{bmatrix} -1 & 4 \\ -2 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = 4t \\ x_2 = t \end{matrix} \rightarrow v_2 = t\begin{bmatrix} 4 \\ 1 \end{bmatrix} \rightarrow \boxed{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}$$

c. Find matrices P and D such that P is nonsingular and $D = P^{-1}AP$ is diagonal.

$$P = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \qquad D = P^{-1}AP = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow \\ v_1 & v_2 \end{matrix} \qquad\qquad \begin{matrix} \uparrow & \uparrow \\ \lambda_1 & \lambda_2 \end{matrix}$$

d. Find A^6.

$$A^6 = PD^6P^{-1} = P\begin{bmatrix} -15625 & 0 \\ 0 & 64 \end{bmatrix}P^{-1} = \begin{bmatrix} -2159 & 8892 \\ -4446 & 17848 \end{bmatrix}$$

3. For the following matrix, $A = \begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$

a. Find all eigenvalues (classify multiplicity if needed).

$0 = \lambda^2 - 0\lambda + 1$

$0 = (\lambda - i)(x + i)$

$$\boxed{\lambda_1 = i \qquad \lambda_2 = -i}$$

b. Find corresponding eigenvectors and bases.

$\lambda I - A = \begin{bmatrix}\lambda - 1 & 1\\ -2 & \lambda + 1\end{bmatrix}$

$\lambda_1: \quad iI - A = \begin{bmatrix}i-1 & 1\\ \boxed{-2 \quad i+1}\end{bmatrix} \rightarrow (-2)x_1 + (i+1)x_2 = 0$ let $x_2 = t$

$(-2)x_1 = -(i+1)t$

$x_1 = \frac{i+1}{2} \longrightarrow t\begin{bmatrix}\frac{i+1}{2}\\ 1\end{bmatrix} \rightarrow \begin{bmatrix}i+1\\ 2\end{bmatrix}$

$$\boxed{\begin{array}{l}\lambda_1 \rightarrow v_1 = \begin{bmatrix}i+1\\ 2\end{bmatrix}\\ \qquad\qquad \text{conjugates}\\ \lambda_2 \rightarrow V_2 = \begin{bmatrix}-i+1\\ 2\end{bmatrix}\end{array}}$$

Proof: $\lambda_2: \quad -iI - A = \begin{bmatrix}-i-1 & 1\\ -2 & -i+1\end{bmatrix}$

$(-2)x_1 + (-i+1)x_2 = 0$ let $x_2 = t$

$(-2)x_1 = -(-i+1)t$

$x_1 = \frac{-i+1}{2}t \longrightarrow t\begin{bmatrix}\frac{-i+1}{2}\\ 1\end{bmatrix} \rightarrow \begin{bmatrix}-i+1\\ 2\end{bmatrix}$

4. For the following matrix, $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda-3 & 1 & -1 \\ -7 & \lambda+5 & -1 \\ -6 & 6 & \lambda-2 \end{bmatrix} \begin{matrix} \lambda-3 & 1 \\ -7 & \lambda+5 \\ -6 & 6 \end{matrix}$$

a. Find all eigenvalues (classify multiplicity if needed).

$$0 = (\lambda-3)(\lambda+5)(\lambda-2) + 6 + 42 - 6(\lambda+5) + 6(\lambda-3) + 7(\lambda-2)$$

$$0 = (\lambda-3)(\lambda+5)(\lambda-2) + 7\lambda - 14$$

$$0 = (\lambda-2)\left[(\lambda-3)(\lambda+5)+7\right] = (\lambda-2)(\lambda^2+2\lambda-8) = (\lambda-2)(\lambda-2)(\lambda+4)$$

$$\boxed{\lambda_1 = -4 \quad \lambda_{2,3} = 2 \leftarrow \text{mult. } 2}$$

b. Find a maximum set of linearly independent eigenvectors of A.

$$-4I - A = \begin{bmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = 0 \\ x_2 = t \\ x_3 = t \end{matrix} \quad t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$2I - A = \begin{bmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = t \\ x_2 = t \\ x_3 = 0 \end{matrix} \quad t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

c. Is A diagonalizable? If yes, find P such that $D = P^{-1}AP$ is diagonal.

$\boxed{\text{No.}}$

Because only 2 eigenvectors can be produced, we can not diagonalize A.

5. Use matrices to solve the following system of differential equations:

$y_1' = -4y_1 + 6y_2$

$y_2' = -3y_1 + 5y_2$ $\quad y_1(0) = 3,\ y_2(0) = 2$

1. $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\qquad y' = Ay$

2. Let $\boxed{y = Pu}$, where P is a matrix that diagonalizes A. Also, let $\boxed{y' = Pu'}$. We don't know what u and u' are, so we will solve for them by substituting.

$y' = Ay$

$Pu' = APu$

$u' = P^{-1}APu$

Because $D = P^{-1}AP$

$u' = Du$

Finding P and D

$0 = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$

$\lambda_1 = 2$ $\qquad \lambda_2 = -1$

$\begin{bmatrix} 6 & -6 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad \begin{bmatrix} 3 & -6 \\ 3 & -6 \end{bmatrix} \to \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ $\qquad D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$u' = nu$

$\downarrow$

$u = ce^{nx}$

RULE (notecard this)

$u_1' = 2u_1$ $\qquad u_2' = -u_2$

$u_1 = C_1 e^{2x}$ $\qquad u_2 = C_2 e^{-x}$

$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} C_1 e^{2x} \\ C_2 e^{-x} \end{bmatrix}$

3. Earlier, we let $y = Pu$: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} C_1 e^{2x} \\ C_2 e^{-x} \end{bmatrix}$

$$\boxed{\begin{aligned} y_1 &= C_1 e^{2x} + 2C_2 e^{-x} \\ y_2 &= C_1 e^{2x} + C_2 e^{-x} \end{aligned}}$$

general sol'n

4. Using our givens $y_1(0) = 3$ and $y_2(0) = 2$:

$3 = C_1 e^0 + 2C_2 e^0$ $\qquad 3 = C_1 + 2C_2$

$2 = C_1 e^0 + C_2 e^0$ $\qquad 2 = C_1 + C_2$

$\Downarrow$

$C_1 = 1$

$C_2 = 1$

$$\boxed{\begin{aligned} y_1 &= e^{2x} + 2e^{-x} \\ y_2 &= e^{2x} + e^{-x} \end{aligned}}$$