Linear Algebra
Exam #3 (Chapter 5)
SPRING 2013
Part 1 – No Calculator

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USE THE METHODS SPECIFIED IN EACH PROBLEM, AND SHOW ALL STEPS TO RECEIVE FULL CREDIT!!

1. Use diagonalization to compute A^{15} for $A = \begin{bmatrix} 4 & -3 \\ 5 & -4 \end{bmatrix}$

$$0 = \lambda^2 - 0\lambda - 1 = (\lambda - 1)(\lambda + 1) \qquad \lambda_1 = 1 \qquad \lambda_2 = -1$$

$$\lambda I - 1 = \begin{bmatrix} \lambda - 4 & 3 \\ -5 & \lambda + 4 \end{bmatrix}$$

$$\lambda_1: \quad I - 1 = \begin{bmatrix} -3 & 3 \\ -5 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = t \\ x_2 = t \end{matrix} \quad t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2: \quad -I - 1 = \begin{bmatrix} -5 & 3 \\ -5 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{3}{5} \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = \frac{3}{5}t \\ x_2 = t \end{matrix} \quad t\begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad A^{15} = PD^{15}P^{-1} = \boxed{\begin{bmatrix} 4 & -3 \\ 5 & -4 \end{bmatrix}}$$

2. For the following vector, $\mathbf{u} = \langle 6,\ 1+4i,\ 6-2i \rangle$, find each of the following:

a. $\overline{\mathbf{u}}$

$\boxed{\langle 6, 1-4i, 6+2i \rangle}$

b. $\mathrm{Re}(\mathbf{u})$

$\boxed{\langle 6, 1, 6 \rangle}$

c. $\mathrm{Im}(\mathbf{u})$

$\boxed{\langle 0, 4, -2 \rangle}$

d. $\|\mathbf{u}\|$

if $u = \langle 6, 1+4i, 6-2i \rangle$:

$\|u\| = \sqrt{6^2 + 1^2 + 4^2 + 6^2 + 2^2} = \boxed{\sqrt{93}}$

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Part 2 – GDC Calculator

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1. Let $x_1(t)$ & $x_2(t)$ be the amounts of two interacting chemicals present together at time t. Suppose they interact according to the differential equations:

$$x_1' = -4x_1 - 2x_2$$
$$x_2' = x_1 - x_2$$

If the initial quantities are $x_1(0) = 9$ & $x_2(0) = 3$, solve the system of differential equations to find the amount of each chemical present after t seconds.

This solution is abbreviated. Please refer to the 2012 test and the guide for an explanation of each step.

$$\boxed{\begin{array}{l} 0 = \lambda^2 + 5\lambda + 6 = (\lambda+2)(\lambda+3) \\ \lambda_1: -2 \quad \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{array}{l} x_1 = -t \\ x_2 = t \end{array} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \lambda_2: -3 \quad \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{array}{l} x_1 = -2t \\ x_2 = t \end{array} \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ P = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \end{array}}$$

① $y' = Ay$ $\quad \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

② $u' = Du$ $\quad \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$u_1' = -2u_1 \rightarrow \quad u_1 = c_1 e^{-2x}$$
$$u_2' = -3u_2 \rightarrow \quad u_2 = c_2 e^{-3x}$$

③ $y = Pu$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-2x} \\ c_2 e^{-3x} \end{bmatrix} \qquad \left.\begin{array}{l} y_1 = -c_1 e^{-2x} - 2c_2 e^{-3x} \\ y_2 = c_1 e^{-2x} + c_2 e^{-3x} \end{array}\right] \text{general sol'n}$$

④ Particular solution

$$y_1(0) = 9 \qquad 9 = -c_1 e^0 - 2c_2 e^0 \qquad 9 = -c_1 - 2c_2 \qquad c_1 = 15$$
$$y_2(0) = 3 \qquad 3 = c_1 e^0 + c_2 e^0 \qquad 3 = c_1 + c_2 \qquad c_2 = -12$$

$$\begin{array}{l} y_1 = -15e^{-2x} + 24e^{-3x} \\ y_2 = 15e^{-2x} - 12e^{-3x} \end{array} \longrightarrow \boxed{\begin{array}{l} x_1 = -15e^{-2t} + 24e^{-3t} \\ x_2 = 15e^{-2t} - 12e^{-3t} \end{array}}$$

I accidentally used y the whole time...

2. For the following matrix, $A = \begin{bmatrix} -7 & -9 & 3 \\ 2 & 4 & -2 \\ -3 & -3 & -1 \end{bmatrix}$ $\quad \lambda I - A = \left[\begin{array}{ccc} \lambda+7 & 9 & -3 \\ -2 & \lambda-4 & 2 \\ 3 & 3 & \lambda+1 \end{array}\right]\begin{array}{cc} \lambda+7 & 9 \\ -2 & \lambda-4 \\ 3 & 3 \end{array}$

a. Find all eigenvalues (classify multiplicity if needed).

$0 = (\lambda+7)(\lambda-4)(\lambda+1) + 54 + 18 + 9(\lambda-4) - 6(\lambda+7) + 18(\lambda+1)$

$0 = (\lambda+7)(\lambda-4)(\lambda+1) + 54 + 18 + 9\lambda - 36 - 6\lambda - 42 + 18\lambda + 18$

$0 = (\lambda+7)(\lambda-4)(\lambda+1) + 21\lambda + 12$

$0 = (\lambda^2+3\lambda-28)(\lambda+1) + 21\lambda + 12$

$0 = \lambda^3 + 3\lambda^2 - 28\lambda + \lambda^2 + 3\lambda - 28 + 21\lambda + 12$

$0 = \lambda^3 + 4\lambda^2 - 4\lambda - 16$

POLYSMLT: $\boxed{\lambda_1 = -4 \quad \lambda_2 = -2 \quad \lambda_3 = 2}$

b. Is A diagonalizable? If yes, find P such that $D = P^{-1}AP$ is diagonal.

A has 3 distinct eigenvalues, so $\boxed{\text{A is diagonalizable.}}$

$\lambda_1: -4I - A = \begin{bmatrix} 3 & 9 & -3 \\ -2 & -8 & 2 \\ 3 & 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} x_1 = t \\ x_2 = 0 \\ x_3 = t \end{array} \quad t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad V_1$

$\lambda_2: -2I - A = \begin{bmatrix} 5 & 9 & -3 \\ -2 & -6 & 2 \\ 3 & 3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} x_1 = 0 \\ x_2 = \frac{1}{3}t \\ x_3 = t \end{array} \quad t\begin{bmatrix} 0 \\ \frac{1}{3} \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \quad V_2$

$\lambda_3: 2I - A = \begin{bmatrix} 9 & 9 & -3 \\ -2 & -2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} x_1 = -t \\ x_2 = t \\ x_3 = 0 \end{array} \quad t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad V_3$

$$\boxed{P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix}}$$

$\uparrow V_1 \quad \uparrow V_2 \quad \uparrow V_3$

$P^{-1}AP = D$

$P^{-1}\begin{bmatrix} -7 & -9 & 3 \\ 2 & 4 & -2 \\ -3 & -3 & -1 \end{bmatrix}P = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ Diagonal ✓

3. Find the eigenvalues and bases for the eigenspaces of $A = \begin{bmatrix} 8 & 6 \\ -3 & 2 \end{bmatrix}$

$$0 = \lambda^2 - 10\lambda + 34$$

POLYSMLT: $\lambda_1 = 5 + 3i$ $\quad\quad \lambda_2 = 5 - 3i$

$$\lambda I - A = \begin{bmatrix} \lambda - 8 & -6 \\ 3 & \lambda - 2 \end{bmatrix}$$

$$\lambda_1 I - A = \begin{bmatrix} -3+3i & -6 \\ 3 & 3+3i \end{bmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} -3+3i & -6 \\ 1 & 1+i \end{bmatrix} \rightarrow x_1 + (1+i)x_2 = 0$$

let $x_2 = t$

$$x_1 = -(1+i)t \longrightarrow t\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$$

$$\lambda_1 \rightarrow v_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$$

conjugates

$$\lambda_2 \rightarrow v_2 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$$

PROOF: $\lambda_2 I - A = \begin{bmatrix} -3-3i & -6 \\ 3 & 3-3i \end{bmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} -3-3i & -6 \\ 1 & 1-i \end{bmatrix} \rightarrow x_1 + (1-i)x_2 = 0$

$$x_2 = t$$

$$x_1 = -(1-i)t$$

$$t\begin{bmatrix} -1+1 \\ 1 \end{bmatrix}$$