Linear Algebra
Exam #3 (Chapter 5)
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USE THE METHODS SPECIFIED IN EACH PROBLEM, AND SHOW ALL STEPS TO RECEIVE FULL CREDIT!!

1. Find the characteristic polynomial of the following matrices:

a. $A = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$

$$\boxed{0 = \lambda^2 - 11\lambda + 24}$$

b. $A = \begin{bmatrix} 9 & 5 & 1 \\ 2 & 0 & 4 \\ 1 & 7 & 6 \end{bmatrix}$

$$\begin{vmatrix} \lambda-9 & -5 & -1 \\ -2 & \lambda & -4 \\ -1 & -7 & \lambda-6 \end{vmatrix} \begin{matrix} \lambda-9 & -5 \\ -2 & \lambda \\ -1 & -7 \end{matrix}$$

$(\lambda-9)(\lambda)(\lambda-6) - 20 - 14 - \lambda - 28(\lambda-9) - 10(\lambda-6)$

$- 20 - 14 - \lambda - 28\lambda + 252 - 10\lambda + 60$

$\lambda^3 - 15\lambda^2 + 54\lambda \; + 278 - 39\lambda$

$$\boxed{0 = \lambda^3 - 15\lambda^2 + 15\lambda + 278}$$

c. $A = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 1 & 5 & 3 & 0 \\ 5 & 4 & 5 & 1 \end{bmatrix}$

$$\boxed{0 = (\lambda-8)(\lambda-2)(\lambda-3)(\lambda-1)}$$

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$40 = 2 \cdot 2 \cdot 2 \cdot 5$

2. For the following matrix, $A = \begin{bmatrix} 6 & 9 \\ 1 & 4 \end{bmatrix}$

a. Find all eigenvalues (classify multiplicity if needed).

$0 = \lambda^2 - 10\lambda + 15$

$\lambda = \dfrac{10 \pm \sqrt{100-60}}{2} = \dfrac{10 \pm \sqrt{40}}{2} = \dfrac{10 \pm 2\sqrt{10}}{2}$

$\boxed{\lambda_1 = 5+\sqrt{10} \quad \lambda_2 = 5-\sqrt{10}}$

b. Find corresponding eigenvectors and bases.

$\lambda I - A = \begin{bmatrix} \lambda-6 & -9 \\ -1 & \lambda-4 \end{bmatrix}$

λ_1: $(5+\sqrt{10})I - A = \begin{bmatrix} -1+\sqrt{10} & -9 \\ -1 & 1+\sqrt{10} \end{bmatrix}$

$(-1+\sqrt{10})x_1 - 9x_2 = 0$

$x_1 = t$

$-9x_2 = (1-\sqrt{10})t$

$x_2 = \dfrac{-1+\sqrt{10}}{9}t$

$V_1 = \begin{bmatrix} 9 \\ -1+\sqrt{10} \end{bmatrix}$

λ_2: $(5-\sqrt{10})I - A = \begin{bmatrix} -1-\sqrt{10} & -9 \\ -1 & 1-\sqrt{10} \end{bmatrix}$

$(-1-\sqrt{10})x_1 - 9x_2 = 0$

$x_1 = t$

$x_2 = \dfrac{\cancel{-}(-1-\sqrt{10})}{\cancel{-}9}t$

$V_2 = \begin{bmatrix} 9 \\ -1-\sqrt{10} \end{bmatrix}$

c. Find matrices P and D such that P is nonsingular and $D = P^{-1}AP$ is diagonal.

$\boxed{P = \begin{bmatrix} 9 & 9 \\ -1+\sqrt{10} & -1-\sqrt{10} \end{bmatrix}}$ $\boxed{D = \begin{bmatrix} 5+\sqrt{10} & 0 \\ 0 & 5-\sqrt{10} \end{bmatrix}}$

d. Find A^8.

$A^k = PD^kP^{-1}$

$A^8 = PD^8P^{-1} = \boxed{\begin{bmatrix} 12965625 & 28035000 \\ 3115000 & 6735625 \end{bmatrix}}$

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3. For the following vector, $\mathbf{u} = \langle 8i,\ 7,\ 7 - 8i \rangle$, find each of the following:

a. $\overline{\mathbf{u}}$

$$\boxed{\bar{u} = \langle -8i, 7, 7+8i \rangle}$$

b. $\text{Re}(\mathbf{u})$

$$\boxed{\text{Re}(u) = \langle 0, 7, 7 \rangle}$$

c. $\text{Im}(\mathbf{u})$

$$\boxed{\text{Im}(u) = \langle 8, 0, -8 \rangle}$$

d. $\|\mathbf{u}\|$

$$\|u\| = \sqrt{8^2 + 7^2 + 7^2 + 8^2} = \boxed{\sqrt{226}}$$

4. Find eigenvalues of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k^3 & -3k^2 & 3k \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -k^3 & 3k^2 & \lambda - 3k \end{bmatrix} \begin{matrix} \lambda & -1 \\ 0 & \lambda \\ -k^3 & 3k^2 \end{matrix}$$

$$\lambda^3 - 3\lambda^2 k - k^3 + 3\lambda k^2 = 0$$

$$(\lambda - k)^3 = 0$$

$(a-b)(a-b)(a-b)$

$(a^2 - 2ab + b^2)(a-b)$

$a^3 - 2a^2 b + ab^2 + 2ab^2 - a^2 b - b^3$

$a^3 - 3a^2 b + 3ab^2 - b^3$

$$\boxed{\lambda_{1,2,3} = k} \leftarrow \text{multiplicity 3}$$

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5. For the following matrix, $A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda-6 & -3 & 8 \\ 0 & \lambda+2 & 0 \\ -1 & 0 & \lambda+3 \end{bmatrix} \begin{matrix} \lambda-6 & -3 \\ 0 & \lambda+2 \\ -1 & 0 \end{matrix}$$

a. Find all eigenvalues (classify multiplicity if needed).

$0 = (\lambda-6)(\lambda+2)(\lambda+3) + 8(\lambda+2)$

$0 = (\lambda+2)\left[\lambda^2 - 3\lambda - 10\right]$

$0 = (\lambda+2)^2(\lambda-5)$

$\lambda_{1,2}$ has multiplicity 2. $\rightarrow$ $\boxed{\lambda_{1,2} = -2 \qquad \lambda_3 = 5}$

b. Find the corresponding eigenvectors of A.

$\lambda_{1,2}$: $-2I - A = \begin{bmatrix} -8 & -3 & 8 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} x_1 = t \\ x_2 = 0 \\ x_3 = t \end{matrix}$ $\quad V_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ☆ No V_2 produced!

λ_3: $5I - A = \begin{bmatrix} -1 & -3 & 8 \\ 0 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} x_1 = 8t \\ x_2 = 0 \\ x_3 = t \end{matrix}$ $\quad V_3 = \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$

c. Is A diagonalizable? If yes, find P such that $D = P^{-1}AP$ is diagonal.

Eigenvalue $\lambda_{1,2} = -2$, with multiplicity 2, did not produce 2 eigenvectors, so P can not be constructed. Thus,

$\boxed{\text{A is not diagonalizable.}}$

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6. Use matrices to solve the following system of differential equations:

$$y_1' = y_1 + 3y_2 \qquad y_1(0) = 5,\ y_2(0) = 6$$
$$y_2' = 2y_1 + 4y_2$$

① $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\lambda^2 - 5\lambda - 2 \qquad \lambda = \frac{5 \pm \sqrt{25+8}}{2} = \frac{5 \pm \sqrt{33}}{2}$

$\lambda I - A = \begin{bmatrix} \lambda-1 & -3 \\ -2 & \lambda-4 \end{bmatrix} \qquad \lambda_1 = \frac{5+\sqrt{33}}{2} \quad \lambda_2 = \frac{5-\sqrt{33}}{2}$

② $u' = Du$

$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = D \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\lambda_1 : \begin{bmatrix} \frac{3+\sqrt{33}}{2} & -3 \\ -2 & \frac{-3+\sqrt{33}}{2} \end{bmatrix} \qquad (3+\sqrt{33})x_1 - 6x_2 = 0 \qquad x_1 = t \qquad x_2 = \frac{3+\sqrt{33}}{6}t \qquad \begin{bmatrix} 6 \\ 3+\sqrt{33} \end{bmatrix}$

$\lambda_2 : \longrightarrow \begin{bmatrix} 6 \\ 3-\sqrt{33} \end{bmatrix}$

$u_1' = \left(\frac{5+\sqrt{33}}{2}\right)u_1 \rightarrow u_1 = c_1 e^{\left(\frac{5+\sqrt{33}}{2}\right)x}$

$u_2' = \left(\frac{5-\sqrt{33}}{2}\right)u_2 \rightarrow u_2 = c_2 e^{\left(\frac{5-\sqrt{33}}{2}\right)x}$

$P = \begin{bmatrix} 6 & 6 \\ 3+\sqrt{33} & 3-\sqrt{33} \end{bmatrix} \qquad D = \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{bmatrix}$

③ $y = Pu = \begin{bmatrix} 6 & 6 \\ 3+\sqrt{33} & 3-\sqrt{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

let $u_1 = c_1 a$ and $u_2 = c_2 b$

$y_1 = 6c_1 a + 6c_2 b \qquad y_1(0) = 5 = 6c_1 + 6c_2 \rightarrow c_2 = \frac{5-6c_1}{6} = \frac{5}{6} - c_1$

$y_2 = (3+\sqrt{33})c_1 a + (3-\sqrt{33})c_2 b \qquad y_2(0) = 6 = (3+\sqrt{33})c_1 + (3-\sqrt{33})c_2$

$6 = (3+\sqrt{33})c_1 - (3+\sqrt{33})c_1 + \frac{15-5\sqrt{33}}{6}$

$6 = 2\sqrt{33}\,c_1 +$ that

$36 = 12\sqrt{33}\,c_1 + 15 - 5\sqrt{33}$

$5\sqrt{33} + 21 = 12\sqrt{33}\,c_1$

$c_1 = \frac{5\sqrt{33}+21}{12\sqrt{33}} \cdot \frac{\sqrt{33}}{\sqrt{33}} = \frac{165+21\sqrt{33}}{396}$

$c_1 = \frac{55+7\sqrt{33}}{132}$

④ $y_1 = \left(\frac{55+7\sqrt{33}}{22}\right) e^{\left(\frac{5+\sqrt{33}}{2}\right)x} + \left(\frac{55-7\sqrt{33}}{22}\right) e^{\left(\frac{5-\sqrt{33}}{2}\right)x}$

$y_2 = \left(\frac{99+19\sqrt{33}}{33}\right) e^{\left(\frac{5+\sqrt{33}}{2}\right)x} + \left(\frac{99-19\sqrt{33}}{33}\right) e^{\left(\frac{5-\sqrt{33}}{2}\right)x}$

$5 = 6\,\frac{55+7\sqrt{33}}{22} + 6c_2$

$110 = 55 + 7\sqrt{33} + 132c_2$

conjugates!

$(3+\sqrt{33})\left(\frac{55+7\sqrt{33}}{132}\right) = \frac{165+76\sqrt{33}+231}{132} = \frac{396+76\sqrt{33}}{132} \qquad \frac{55-7\sqrt{33}}{132} = c_2$

$(3-\sqrt{33})\left(\frac{55-7\sqrt{33}}{132}\right) = 165 - 76\sqrt{33} + 231$

$(3+\sqrt{33})(55-7\sqrt{33}) = 165 \cancel{-21\sqrt{33}+55\sqrt{33}} - 231$

$+\ 34$

$33 \quad 17$

$= -66 + 34\sqrt{33}$

$\cancel{22}$

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$(3-\sqrt{33})(55+7\sqrt{33})$

$165 + 21\sqrt{33} - 55\sqrt{33} - 231$

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BONUS: Given vectors $\mathbf{u} = \langle i,\ 2i,\ 3\rangle$, $\mathbf{v} = \langle 4, -2i,\ 1+i\rangle$, $\mathbf{w} = \langle 2-i,\ 2i,\ 5+3i\rangle$

Compute $\overline{(\mathbf{u}\cdot\bar{\mathbf{v}}) - (\bar{\mathbf{w}}\cdot\mathbf{u})}$ why tho...

a) $u\cdot\bar{v}$) $(i)(4)+(2i)(2i)+(3)(1-i) = -1+i$

b) $w\cdot u$) $(2-i)(i)+(2i)(2i)+(5+3i)(3) = 12+11i$

c) $\overline{w\cdot u}$) $= 12-11i$

$a - c = -13+12i \Longrightarrow \boxed{-13-12i}$