Linear Algebra
Exam #4 (Chapter 6)
SPRING 2013

Name Jacob Magnuson **Date** 2016-12-07 **Period** 7

SHOW ALL STEPS TO RECEIVE FULL CREDIT!! Graphing calculators are permitted.

1. **Suppose that the following table shows the annual sales, in millions of dollars, for Advanced Auto Parts and Auto Zone for 2000 through 2007. Find the least squares quadratic model for Advanced Auto Parts. Let *t* represents the year, with *t* = 0 corresponding to 2000. Round all coefficients to two decimal places.**

	Year	Advanced Auto Parts
$x=0$	2000	2564
$x=1$	2001	1635
	2002	1044
	2003	1245
	2004	1376
	2005	2064
	2006	4215
$x=7$	2007	6505

$$p(x) = a_0 x^0 + a_1 x^1 + a_2 x^2$$

We will solve $A^T A \vec{x} = A^T b$:

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \\ 1 & 7 & 49 \end{bmatrix} \\ \begin{array}{ccc} x^0 & x^1 & x^2 \end{array} \end{array} \qquad b = \begin{bmatrix} 2564 \\ 1635 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 6505 \end{bmatrix}$$

$$\vec{x} = (A^T A)^{-1} A^T b = \begin{bmatrix} 2715.42 \\ -1434.36 \\ 279.19 \end{bmatrix} \begin{array}{l} \leftarrow a_0 \\ \leftarrow a_1 \\ \leftarrow a_2 \end{array}$$

$$\boxed{y = 2715.42 - 1434.36x + 279.19x^2}$$

2. **Transform the basis $B = \{\mathbf{u_1} = \langle 0,9,18\rangle, \mathbf{u_2} = \langle 18,0,0\rangle, \mathbf{u_3} = \langle 1,1,1\rangle$ for $\mathbb{R}^3$ into an orthonormal basis. Use the Euclidean inner product for $\mathbb{R}^3$ and use the vectors in the order in which they are shown.**

Since we are given a basis, we are allowed to scale each item as desired. To make computation easier, we will scale the vectors in B as follows:

$$B = \{u_1 = \langle 0,1,2\rangle, u_2 = \langle 1,0,0\rangle, u_3 = \langle 1,1,1\rangle\}$$

<u>Graham-Schmidt process</u>

$V_1 = u_1 = \langle 0,1,2\rangle \qquad \|v_1\| = \sqrt{5}$

$V_2 = u_2 - \dfrac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$

$= \langle 1,0,0\rangle - \dfrac{0}{5}\langle 0,1,2\rangle = \langle 1,0,0\rangle \qquad \|v_2\| = 1$

$V_3 = u_3 - \dfrac{\langle u_3, v_2\rangle}{\|v_2\|^2} V_2 - \dfrac{\langle u_3, v_1\rangle}{\|v_1\|^2} V_1$

$= \langle 1,1,1\rangle - \dfrac{1}{1}\langle 1,0,0\rangle - \dfrac{3}{5}\langle 0,1,2\rangle$

$= \langle 1-1, 1-\frac{3}{5}, 1-\frac{6}{5}\rangle = \underbrace{\langle 0, \tfrac{2}{5}, \tfrac{-1}{5}\rangle}_{\times 5} = \langle 0,2,-1\rangle \qquad \|v_3\| = \sqrt{5}$

Note: because we are still working with a set of basis vectors, we can apply scalars and still have orthogonal vectors.

Finally, divide each orthogonal vector by its norm to produce the orthonormal basis:

$$\{\langle 0, \tfrac{1}{\sqrt{5}}, \tfrac{2}{\sqrt{5}}\rangle, \langle 1,0,0\rangle, \langle 0, \tfrac{2}{\sqrt{5}}, \tfrac{-1}{\sqrt{5}}\rangle\}$$

3. Classify the following sets of vectors as orthogonal (only), orthonormal, or neither.

a. $\{\langle 1,-1\rangle, \langle -1,1\rangle\}$

$\quad u_1 \qquad u_2$

$\langle u_1, u_2\rangle = 1(-1) + (-1)(1) = -2$ ← not orthogonal

$\therefore$ $\boxed{\text{NEITHER}}$

b. $\{\langle 4,-1,1\rangle, \langle -4,-17,-1\rangle, \langle -1,0,4\rangle\}$

$\quad u_1 \qquad u_2 \qquad u_3$

$$\left.\begin{aligned}\langle u_1, u_2\rangle &= (4)(-4) + (-1)(-17) + (1)(-1) = 0 \checkmark\\ \langle u_2, u_3\rangle &= (-4)(-1) + (-17)(0) + (-1)(4) = 0 \checkmark\\ \langle u_1, u_3\rangle &= (4)(-1) + (-1)(0) + (1)(4) = 0 \checkmark\end{aligned}\right\} \text{orthogonal}$$

$\underbrace{\|u_1\| = \sqrt{18} \neq 1}_{\text{not orthonormal}}$

$\therefore$ $\boxed{\text{ORTHOGONAL}}$

c. $\{\langle \frac{\sqrt{2}}{5}, 1, \frac{\sqrt{2}}{5}\rangle, \langle -\frac{\sqrt{6}}{8}, \frac{\sqrt{6}}{4}, \frac{\sqrt{6}}{8}\rangle, \langle \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{-\sqrt{3}}{4}\rangle\}$

$\quad u_1 \qquad u_2 \qquad u_3$

$$\langle u_1, u_2\rangle = \left(\frac{\sqrt{2}}{5}\right)\left(-\frac{\sqrt{6}}{8}\right) + (1)\left(\frac{\sqrt{6}}{4}\right) + \left(\frac{\sqrt{2}}{5}\right)\left(\frac{\sqrt{6}}{8}\right) = -\frac{\sqrt{12}}{40} + \frac{\sqrt{6}}{4} + \frac{\sqrt{12}}{40} = \frac{\sqrt{6}}{4} \neq 0$$

↑ not orthogonal

$\therefore$ $\boxed{\text{NEITHER}}$

4. Use the functions $f(x) = 2$ and $g(x) = 5x^4 - 1$ in $C[-1,1]$ to find $\langle f, g\rangle$ for the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$

$$f(x)g(x) = (2)(5x^4 - 1) = 10x^4 - 2$$

$$\int_{-1}^{1} f(x)g(x)dx = \int_{-1}^{1} 10x^4 - 2\, dx = 2x^5 - 2x\Big|_{-1}^{1} = (2(1)^5 - 2(1)) - (2(-1)^5 - 2(-1))$$

$$= \boxed{0}$$

5. **Given the following matrices:** $A = \begin{bmatrix} 4 & 2 \\ 3 & 2 \\ 5 & 3 \\ 2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 2 \end{bmatrix}$

find the least squares approximation of the system $A\mathbf{x} = \mathbf{b}$, as well as the associated error.

$$\vec{x} = (A^T A)^{-1} A^T b = \boxed{\begin{bmatrix} 12/11 \\ -6/11 \end{bmatrix}} \leftarrow \text{Least Squares Approximation}$$

$$\vec{b} - A\vec{x} = \boxed{\begin{bmatrix} -3/11 \\ -2/11 \\ 2/11 \\ 4/11 \end{bmatrix}} \qquad \|b - A\vec{x}\| \approx \boxed{0.522}$$

error vector

error

6. Given the vectors **u** = (0, 4, 7, 3) and **v** = (2, -1, 5, 2):

a. Find the angle in radians between **u** & **v**.

$$\langle u, v\rangle = (0)(2) + (4)(-1) + (7)(5) + (3)(2) = 37$$

$$\|u\| = \sqrt{0+16+49+9} = \sqrt{74} \qquad \|v\| = \sqrt{4+1+25+4} = \sqrt{34}$$

$$\theta = \cos^{-1}\left(\frac{\langle u,v\rangle}{\|u\|\|v\|}\right) = \cos^{-1}\left(\frac{37}{\sqrt{74}\sqrt{34}}\right) = \boxed{0.741 \text{ rad.}}$$

b. Find d (**u**, **v**) for the standard Euclidean inner product defined in $\mathbb{R}^4$.

$$d(u,v) = \|u - v\| = \|\langle -2, 5, 2, 1\rangle\| = \sqrt{4+25+4+1} = \boxed{\sqrt{34}}$$

c. Find $\langle \mathbf{u}, \mathbf{v}\rangle$ for the inner product $\langle \mathbf{u}, \mathbf{v}\rangle = 4u_1v_1 + 3u_2v_2 + 2u_3v_3 + 1u_4v_4$ defined in $\mathbb{R}^4$.

Just plug in: $\langle u,v\rangle = 4(0)(2) + 3(4)(-1) + 2(7)(5) + 1(3)(2)$

$$= \boxed{64}$$

d. Find $\|\mathbf{u}\|$ for the inner product from *part c* defined in $\mathbb{R}^4$.

$\|u\| = \sqrt{\langle u,u\rangle}$ ← we will use the inner product from c:

$$= \sqrt{4u_1u_1 + 3u_2u_2 + 2u_3u_3 + 1u_4u_4} = \sqrt{0+48+98+9}$$

$$= \boxed{\sqrt{155}}$$