Linear Algebra
Exam #4 (Chapter 6)
FALL 2016

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SHOW ALL STEPS TO RECEIVE FULL CREDIT!! Graphing calculators are permitted.

1. The following table shows Mr. Oddi's monthly bill from FPL, for December 2015 through November 2016. Find the least squares quadratic model for the monthly bill. Let *t* represent the month, with *t* = 0 corresponding to December 2015. Round all coefficients to two decimal places.

t	Month	Bill
0	December 2015	$122.23
1	January 2016	$97.46
2	February 2016	$79.49
3	March 2016	$87.31
4	April 2016	$94.98
5	May 2016	$133.02
6	June 2016	$193.53
7	July 2016	$243.77
8	August 2016	$262.28
9	September 2016	$189.93
10	October 2016	$164.67
11	November 2016	$105.31

= b

I entered 112.23 like an idiot...

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \\ 1 & 7 & 49 \\ 1 & 8 & 64 \\ 1 & 9 & 81 \\ 1 & 10 & 100 \\ 1 & 11 & 121 \end{bmatrix}$$

$$p(t) = a_0 t^0 + a_1 t^1 + a_2 t^2$$

$$A \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = b \qquad \vec{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix} 56.89 \\ 32.48 \\ -2.1 \end{bmatrix} \begin{matrix} a_0 \\ a_1 \\ a_2 \end{matrix}$$

↑ $\vec{x}$

−2

$$\boxed{p(t) = 56.89 + 32.48t - 2.1t^2}$$

BONUS: Would the least squares cubic model provide a more accurate approximation? Provide a numerical justification for your answer.

Quadratic error:

$$\|\vec{b} - A\vec{x}\| = \left\| \begin{bmatrix} 55.34 \\ 10.19 \\ -33.97 \\ -46.13 \\ -58.24 \\ -33.79 \\ 17.34 \\ 62.396 \\ 79.92 \\ 10.79 \\ -7.05 \\ -64.79 \end{bmatrix} \right\| = 157.86$$

↑ [I]

Cubic: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ & A_{quad} & & \vdots \\ 1 & 11 & 121 & 1331 \end{bmatrix}$ ↑ t^3 $\vec{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix} 123.39 \\ -60.89 \\ 20.07 \\ -1.34 \end{bmatrix}$

+2

$$\text{error} = \|\vec{b} - A\vec{x}\| = \left\| \begin{bmatrix} -11.16 \\ 16.23 \\ 8.36 \\ 2.25 \\ -19.68 \\ -19.96 \\ 3.23 \\ 24.11 \\ 29.54 \\ -31.53 \\ -13.1 \\ 11.71 \end{bmatrix} \right\| = 63.37$$

↑ [J]

err(quad) > err(cubic)
∴ cubic LSS is a more accurate model.

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2. For the following basis $B = \{\mathbf{u}_1 = \langle 1,1,1\rangle, \mathbf{u}_2 = \langle -2,0,4\rangle, \mathbf{u}_3 = \langle 0,3,6\rangle\}$ in $\mathbb{R}^3$:

a. Transform B into an orthoGONal basis. Use the Euclidean inner product for $\mathbb{R}^3$ and use the vectors in the order in which they are shown.

$B = \{\langle 1,1,1\rangle, \langle -1,0,2\rangle, \langle 0,1,2\rangle\}$?
$u_1 \quad u_2 \quad u_3$

$v_1 = u_1 = \langle 1,1,1\rangle \quad \|v_1\| = \sqrt{3}$

OK ✓

$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$

$= \langle -1,0,2\rangle - \frac{1}{3}\langle 1,1,1\rangle = \langle -\frac{4}{3}, -\frac{1}{3}, \frac{5}{3}\rangle \cdot 3 = \langle -4,-1,5\rangle$

$\|v_2\| = \sqrt{42}$

$v_3 = u_3 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1$

$= \langle 0,1,2\rangle - \frac{9}{42}\langle -4,-1,5\rangle - \frac{3}{3}\langle 1,1,1\rangle = \langle -\frac{1}{7}, \frac{3}{14}, -\frac{1}{14}\rangle \cdot 14$

$= \langle -2, 3, -1\rangle \quad \|v_3\| = \sqrt{14}$

$(v_1, v_2) = 0$ ✓

$\langle v_2, v_3\rangle = 0$ ✓

$\langle v_1, v_3\rangle = 0$ ✓

Orthogonal basis
$\{v_1 = \langle 1,1,1\rangle, v_2 = \langle -4,-1,5\rangle, v_3 = \langle -2,3,-1\rangle\}$

b. Using your answer to part (a), transform B into an orthoNORMal basis.

$\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\}$ FT OK ✓

orthonormal basis

$\{\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle, \langle -\frac{4}{\sqrt{42}}, -\frac{1}{\sqrt{42}}, \frac{5}{\sqrt{42}}\rangle, \langle -\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}\rangle\}$

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3. Classify the following sets of vectors as orthogonal (only), orthonormal, or neither.

a. $\{\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle\}$

$\hat{i} \quad \hat{j} \quad \hat{k}$

$\langle \hat{i},\hat{j}\rangle = \langle \hat{j},\hat{k}\rangle = \langle \hat{i},\hat{k}\rangle = 0 \quad \therefore$ orthogonal

$\|\hat{i}\| = \|\hat{j}\| = \|\hat{k}\| = 1 \quad \therefore$ all unit vectors

$\therefore$ **ORTHONORMAL**

b. $\{\langle 2,8,0\rangle, \langle 0,0,4\rangle, \langle -2,\frac{1}{2},0\rangle\}$

a b c

$\langle a,b\rangle = 0$ ✓

$\langle b,c\rangle = 0$ ✓

$\langle a,c\rangle = (2)(-2) + (8)(\frac{1}{2}) + 0 = -4+4 = 0$ ✓

} $\therefore$ orthogonal

$\|a\| = \sqrt{68} \neq 1 \quad \therefore$ not unit vector

$\therefore$ **ORTHOGONAL** (only)

4. Use the functions $f(x) = x+3$ and $g(x) = 5x$ in $C[-1,1]$ to find $\langle f,g\rangle$ for the inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)dx$

$f(x)g(x) = 5x^2 + 15x = 5(x^2+3x)$

$\langle f,g\rangle = 5\int_{-1}^{1} x^2+3x \, dx = 5\left(\frac{1}{3}x^3 + \frac{3}{2}x^2 \Big|_{-1}^{1}\right) = 5\left[\left(\frac{1}{3}+\frac{3}{2}\right) - \left(-\frac{1}{3}+\frac{3}{2}\right)\right] = \boxed{\frac{10}{3}}$

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5. Given the vectors $\mathbf{u} = (0, 5, 2, 4)$ and $\mathbf{v} = (2, 0, 1, 7)$:

a. Find the angle in radians between **u** & **v**.

$$\theta = \cos^{-1}\left(\frac{\langle u, v\rangle}{\|u\|\|v\|}\right)$$

$$= \cos^{-1}\left(\frac{30}{\sqrt{45}\cdot\sqrt{54}}\right)$$

$$\langle u,v\rangle = (2)(1)+(4)(7) = 30$$

$$\|u\| = \sqrt{25+4+16} = \sqrt{45}$$

$$\|v\| = \sqrt{4+1+49} = \sqrt{54}$$

$\theta = 0.9165$ rad.

b. Find $d(\mathbf{u}, \mathbf{v})$ for the standard Euclidean inner product defined in $\mathbb{R}^4$.

$$d(u,v) = \|u-v\| = \|\langle -2, 5, 1, -3\rangle\| = \sqrt{\langle w, w\rangle} = \sqrt{4+25+1+9} = \sqrt{39}$$

($\vec{w} = \langle -2, 5, 1, -3\rangle$)

c. Find $\langle \mathbf{u}, \mathbf{v}\rangle$ for the inner product $\langle \mathbf{u}, \mathbf{v}\rangle = -u_1v_1 + 2u_2v_2 - u_3v_3 + 2u_4v_4$ defined in $\mathbb{R}^4$.

$$\langle u,v\rangle = -(0)(2) + 2(5)(0) - (2)(1) + 2(4)(7) = 54$$

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